

RELATIVITY: THE GENERAL THEORY

BY

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1960

NORTH-HOLLAND PUBLISHING COMPANY, AMSTERDAM

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PUBLISHERS: NORTH-HOLLAND PUBLISHING COMPANY, AMSTERDAM
SOLE DISTRIBUTORS FOR U.S.A.:
INTERSCIENCE PUBLISHERS INC., NEW YORK

PRINTED IN THE NETHERLANDS

*To my friends J. P. and J. J.
who bought nothing half so precious
as what they sold*

PREFACE

Of all physicists, the general relativist has the least social commitment. He is the great specialist in gravitational theory, and gravitation is socially significant, but he is not consulted in the building of a tower, a bridge, a ship, or an aeroplane, and even the astronauts can do without him until they start wondering which ether their signals travel in.

Splitting hairs in an ivory tower is not to everyone's taste, and no doubt many a relativist looks forward to the day when governments will seek his opinion on important questions. But what does 'important' mean? Science has a dual aim, to understand nature and to conquer nature, but in the intellectual life of man surely it is the understanding which is the more important. Then let the relativist rejoice in the ivory tower where he has peace to seek understanding of Einstein's theory as long as the busy world is satisfied to do its jobs without him. Let him be satisfied with the difficult task of gaining for himself as much understanding as he can, and the still more difficult task of transmitting to others the scraps of understanding he has been able to gain.

It is hard for an author to detach himself from his book and view it objectively. But it seems to me that the spirit of this book is best described by the word *irony*, used in the sense of the *εἰρωνεία* of Socrates, and that sense is not at all easy to explain. There are heavy calculations in the book, but there are places where the reader will find me sitting on the fence, whistling, instead of rushing into the fray. Would it not be better to get *something* done, even though one might not quite understand *what*? But that is precisely what irony forbids. The lust for calculation must be tempered by periods of inaction, in which the mechanism is completely unscrewed and then put together again. It is the decarbonization of the mind. It would perhaps seem that irony of this sort belongs to evaluation rather than to creation, but in the whole history of science there is no greater example of irony than when Einstein said he did not know what absolute time was, a thing which everyone knew.

For the purposes of this book, the general theory of relativity means that theory of gravitation to which Einstein gave definite form in 1916. But there is no attempt to give a survey of all the work done in that field in the past forty-odd years. Rather the book attempts to combine an account of a selection of outstanding topics with the development of a new method. This method is based on a certain function introduced into tensor calculus by H. S. Ruse nearly thirty years ago, but since that time hardly used. This function (called *world-function* here) is, to within a trivial factor, the square of the geodesic distance between two events in space-time, regarded as a function of their eight coordinates. It proves a powerful tool in calculations, since the usual method of approximation by power series may be used without abandoning the facilities of tensor calculus.

In compensation for lack of completeness in the coverage of the subject, there is a fairly extensive Bibliography. To keep this from getting altogether too large, it was necessary to select, and for any omissions of important references which should have been included I offer a sincere apology in advance. To increase the utility of the Bibliography, reviews are cited in almost all cases. If such a Bibliography were expanded into one or two volumes by including not merely the references to reviews but the reviews themselves, the result would be most useful to students of relativity.

I have described the spirit of this book as ironical, but it is also geometrical. Perhaps the two things go together, for a simple space-time diagram will often bring out the inner meaning of a mass of calculations. Surely one of the reasons why the general theory of relativity remains a mystery to so many physicists is that they do not realize how easy it is to form a qualitative geometrical image of what is going on. It is in fact easier to deal with space-time diagrams, which remain fixed, than with the kinematical pictures of Newtonian mechanics.

If we accept the idea that space-time is a Riemannian four-space (and if we are relativists we must), then surely our first task is to get the feel of it just as early navigators had to get the feel of a spherical ocean. And the first thing we have to get the feel of is the Riemann tensor, for it *is* the gravitational field — if it vanishes, and only then, there is no field. Yet, strangely enough, this most important element has been pushed into the background. When I started to write this book, I did not know what were the values in my study of the twenty

invariant components of the Riemann tensor, nor did I know the value of the first curvature of my world-line, let alone the second and third. I know more now, and what I know I have put into the book. I know now that if I break my neck by falling off a cliff, my death is not to be blamed on the force of gravity (what does not exist is necessarily guiltless), but on the fact that I did not maintain the first curvature of my world-line, 'exchanging its security for a dangerous geodesic. To the ironical mind there is little distinction between the mundane and the exalted, and that is no doubt why Socrates had to drink the hemlock cup.

I am much indebted to the well known books of Pauli, Eddington, Tolman, Bergmann, Møller and Lichnerowicz, but the geometrical way of looking at space-time comes directly from Minkowski. He protested against the use of the word 'relativity' to describe a theory based on an 'absolute' (space-time), and, had he lived to see the general theory of relativity, I believe he would have repeated his protest in even stronger terms. However, we need not bother about the name, for the word 'relativity' now means primarily Einstein's theory and only secondarily the obscure philosophy which may have suggested it originally. It is to support Minkowski's way of looking at relativity that I find myself pursuing the hard path of the missionary. When, in a relativistic discussion, I try to make things clearer by a space-time diagram, the other participants look at it with polite detachment and, after a pause of embarrassment as if some childish indecency had been exhibited, resume the debate in their own terms. Perhaps they speak of the Principle of Equivalence. If so, it is my turn to have a blank mind, for I have never been able to understand this Principle. Does it mean that the signature of the space-time metric is $+2$ (or -2 if you prefer the other convention)? If so, it is important, but hardly a Principle. Does it mean that the effects of a gravitational field are indistinguishable from the effects of an observer's acceleration? If so, it is false. In Einstein's theory, either there is a gravitational field or there is none, according as the Riemann tensor does not or does vanish. This is an absolute property; it has nothing to do with any observer's world-line. Space-time is either flat or curved, and in several places in the book I have been at considerable pains to separate truly gravitational effects due to curvature of space-time from those due to curvature of the observer's world-line (in most ordinary cases the latter predominate). The Principle of Equivalence

performed the essential office of midwife at the birth of general relativity, but, as Einstein remarked, the infant would never have got beyond its long-clothes had it not been for Minkowski's concept. I suggest that the midwife be now buried with appropriate honours and the facts of absolute space-time faced.

Attention is drawn to the Appendixes at the end of the book. Appendix A explains the notation, with a polemic against certain conventions (mere trivialities, of course), while Appendix B contains a modest proposal for the names of multiples and submultiples of units, and also a list of physical quantities, all expressed in seconds; I have found this list very useful for a rapid comparison of magnitudes.

For a number of years, at the University of Toronto, the patience of a graduate class permitted me to get some understanding of relativity by lecturing on it, and that understanding has been increased and consolidated, during the past eleven years, by many seminars and informal discussions in the School of Theoretical Physics at the Dublin Institute for Advanced Studies. In particular, I owe much to Professors C. Lanczos and E. Schrödinger, in spite of (or perhaps because of) the fact that our points of view have often differed. The Scholars of our School, coming from various countries with a wide variety of background, have been such a powerful stimulus that I feel I have stolen from them much more than I have been able to give. Work by Balazs, Bass, Bertotti, Das, Israel, Mast, O'Brien, O'Raifeartaigh, Pirani, Rayner and Strathdee is cited in the Bibliography; in writing this book, I believe that I have subconsciously (even at times consciously) had these Scholars, past and present, in mind as an unofficial Board of Censors to eliminate obscurity and nonsense.

I thank Mr. A. Das and Dr. F. A. E. Pirani for their labours in reading the proofs of the book, but my gratitude extends beyond that. In many discussions they helped me to explore regions of which I was partially or completely ignorant. Thanks to Mr. Das (and in some measure to Dr. W. B. Bonnor also) I was encouraged to lay aside for a while the garment of geometry (what you see makes sense) and wrestle naked with the formalism of axial symmetry and those universes here called electrovac, both offspring of the subtle mind of Hermann Weyl. Dr. Pirani introduced me to the transport law of Fermi which plays an important part in the book, and my attempt to turn Riemannian geometry into observational physics (measure the Riemann tensor!)

originated largely in discussions with him, with further developments in discussions with Dr. C. B. Mast. Dr. Pirani's comments on the book when in proof have been most helpful, but to him and to Mr. Das I give the usual clearance: all the errors in the book are mine.

Dublin, 1960

J. L. S.

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CHAPTER I

ESSENTIAL TENSOR FORMULAE FOR RIEMANNIAN SPACE-TIME

§ 1. THE METRIC TENSOR AND ADMISSIBLE COORDINATES

A knowledge of tensor calculus is assumed, but it is convenient to set down in this first chapter some essential tensor formulae and developments from them. The next chapter is devoted to a rather novel tensorial technique (world-function), and physical ideas do not appear until Chapter III. Some readers may prefer to start there, and refer back to the first two chapters as the occasion arises.

We have before us a Riemannian space of four dimensions, to which we give the name *space-time*. Using Latin suffixes ¹ for the range of values 1, 2, 3, 4, with the usual summation convention, we denote the coordinates by x^i and the *metric* (or *fundamental*) *tensor* by g_{ij} ($= g_{ji}$). The invariant

$$\Phi = g_{ij}dx^i dx^j \tag{1}$$

is the *metric* (or *fundamental*) *form*. It has the signature $+ 2$, which means that, at any selected point, g_{ij} is reducible to the diagonal matrix (1, 1, 1, $- 1$).

How smooth are the ten functions g_{ij} ? In physics we usually lay such questions aside until they are forced on us, and we would be inclined to accept extreme smoothness (i.e. infinite differentiability). However it is convenient to adopt for the sun, the earth, or other body, a model in which there is an abrupt change from matter to vacuum, and some discontinuity in the smoothness of g_{ij} will occur as a result of this abrupt change. Some hypothesis must be made to cover such a situation, and we shall here follow LICHNEROWICZ [1955a].

We assume that space-time can be broken up into overlapping domains, with a system of *admissible coordinates* x^i in each domain,

¹ See Appendix A for remarks on notation, which is in general that of SYNGE and SCHILD [1956]. All references are to the Bibliography at the end of the book.

and C^3 transformations¹ between the overlapping coordinates. But this is not the important thing — we may be able to cover all space-time with a single system of admissible coordinates. The essential thing is that each domain of the admissible coordinates may happen to be divided into a number of subdomains by *3-spaces of discontinuity*. Over the whole domain, g_{ij} are only C^1 , but in the subdomains they are C^3 ; thus we expect discontinuities in the second derivatives of g_{ij} when we cross a 3-space of discontinuity. This may look a very artificial assumption. The justification is that it fits in with physical ideas when we apply it, and (although this is a somewhat dubious justification) it is analogous to what we have in the theory of potential if we regard the ten quantities g_{ij} as analogues of the Newtonian potential.

The reader is advised not to bother about the above refinement until consideration of discontinuities is forced upon him, but rather to assume in what follows that g_{ij} may be differentiated as often as we please.

For any contravariant vector V^i , the quadratic form $g_{ij}V^iV^j$ is positive, negative or zero. If the value is not zero, we define the *indicator* of V^i , denoted by $\varepsilon(V)$, to be ± 1 so as to make

$$\varepsilon(V)g_{ij}V^iV^j > 0. \quad (2)$$

We use the following terminology:

$$\begin{aligned} g_{ij}V^iV^j < 0, \quad \varepsilon(V) &= -1, \text{ the vector is timelike;} \\ g_{ij}V^iV^j > 0, \quad \varepsilon(V) &= 1, \text{ the vector is spacelike;} \\ g_{ij}V^iV^j = 0, \text{ the vector is null.} \end{aligned}$$

In the case of a null vector it is sometimes convenient to assign an indicator ± 1 ; it does not matter which value we choose, because it is multiplied by zero.

An infinitesimal vector dx^i at a point x^i has a *magnitude* or *norm*

$$ds = (\varepsilon g_{ij}dx^i dx^j)^{\frac{1}{2}} \geq 0, \quad (3)$$

where ε is the indicator of dx^i . We may also call it the *infinitesimal measure* of the vector dx^i . For any curve C joining points A and B , the

¹ Transformations with continuous third derivatives.

integral

$$L = \int_A^B ds = \int_A^B (\varepsilon g_{ij} dx^i dx^j)^{\frac{1}{2}} \quad (4)$$

defines a *finite measure*¹, which of course depends on the curve C .

Some well known formulae are listed here for convenience of reference (commas indicate partial derivatives):

$$\text{Kronecker delta} = \delta_j^i = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \quad (5)$$

$$g = \det g_{ij} < 0. \quad (6)$$

$$g^{ij}g_{ik} = \delta_k^j. \quad (7)$$

$$\begin{aligned} \text{Christoffel symbols: } [ij, k] &= \frac{1}{2}(g_{ik,j} + g_{jk,i} - g_{ij,k}), \\ \Gamma_{jk}^i &= \{j^i_k\} = g^{ia}[jk, a], \\ \Gamma_{ai}^a &= \frac{1}{2}[\log(-g)]_{,i} = \frac{1}{2}g^{-1}g_{,i}, \\ g_{ij,k} &= g_{ia}\Gamma_{jk}^a + g_{ja}\Gamma_{ik}^a. \end{aligned} \quad (8)$$

Given any point² P , it is possible to choose coordinates such that at P

$$g_{ij,k} = 0, \quad [ij, k] = 0, \quad \Gamma_{jk}^i = 0. \quad (9)$$

This greatly reduces the complexity of certain algebraic computations.

§ 2. DERIVATIVES AND GEODESICS

For vector and tensor fields defined throughout a domain of space-time, the *covariant derivatives* (indicated by a vertical stroke) are as follows:

$$V_{|j}^i = V_{,j}^i + \Gamma_{ja}^i V^a, \quad (10)$$

$$V_{i|j} = V_{i,j} - \Gamma_{ij}^a V_a, \quad (11)$$

$$T_{|k}^{ij} = T_{,k}^{ij} + \Gamma_{ka}^i T^{aj} + \Gamma_{ka}^j T^{ia}, \quad (12)$$

$$T_{j|k}^i = T_{j,k}^i + \Gamma_{ka}^i T_j^a - \Gamma_{jk}^a T_a^i, \quad (13)$$

$$T_{ij|k} = T_{ij,k} - \Gamma_{ik}^a T_{aj} - \Gamma_{jk}^a T_{ia}, \quad (14)$$

¹ Abstract geometry borrows words from elementary physical geometry, and this leads to considerable semantic confusion. The geometer is inclined to use the word *length* for the integral L . But this is dangerous, because the word *length* has already got some sort of physical meaning which may, or may not, agree with this mathematical definition of the term. We shall avoid the word *length* here, and introduce it only in Chapter III with suitable caution.

² It is also possible to satisfy (9) along a given curve or, under certain conditions, on a given subspace; cf. FERMI [1922], O'RAIFEARTAIGH [1958b].

with similar formulae for tensors of higher orders. We have identically

$$g_{ij|k} = 0, \quad \delta_{j|k}^i = 0, \quad g^{ij}_{|k} = 0. \quad (15)$$

The covariant derivative of a scalar is the partial derivative.

For vector and tensor fields defined on a curve $x^i = x^i(u)$, the *absolute derivatives*¹ are as follows:

$$\frac{\delta V^i}{\delta u} = \frac{dV^i}{du} + \Gamma_{ab}^i V^a \frac{dx^b}{du}, \quad (16)$$

$$\frac{\delta V_i}{\delta u} = \frac{dV_i}{du} - \Gamma_{ib}^a V_a \frac{dx^b}{du}, \quad (17)$$

$$\frac{\delta T^{ij}}{\delta u} = \frac{dT^{ij}}{du} + \Gamma_{ab}^i T^{aj} \frac{dx^b}{du} + \Gamma_{ab}^j T^{ia} \frac{dx^b}{du}, \quad (18)$$

$$\frac{\delta T_j^i}{\delta u} = \frac{dT_j^i}{du} + \Gamma_{ab}^i T_j^a \frac{dx^b}{du} - \Gamma_{jb}^a T_a^i \frac{dx^b}{du}, \quad (19)$$

$$\frac{\delta T_{ij}}{\delta u} = \frac{dT_{ij}}{du} - \Gamma_{ib}^a T_{aj} \frac{dx^b}{du} - \Gamma_{jb}^a T_{ia} \frac{dx^b}{du}, \quad (20)$$

with similar formulae for tensors of higher orders. We have identically

$$\frac{\delta}{\delta u} g_{ij} = 0, \quad \frac{\delta}{\delta u} \delta_j^i = 0, \quad \frac{\delta}{\delta u} g^{ij} = 0. \quad (21)$$

The absolute derivative of a scalar is the ordinary derivative.

For both covariant and absolute differentiation, the ordinary rule for differentiating a product holds. This important fact may be exhibited in brief form, the suffixes on the tensors A and B being suppressed:

$$(AB)_{|i} = A_{|i}B + AB_{|i}, \quad (22)$$

$$\frac{\delta}{\delta u} (AB) = \frac{\delta A}{\delta u} B + A \frac{\delta B}{\delta u}. \quad (23)$$

¹ It is a curious fact, historically, that the formulae (10)–(15) occupy a more prominent position in works on relativity than the much more powerful formulae (16)–(21). However we shall make little use of either the one or the other in explicit form. Our aim should be to work as far as possible in tensors, and the Γ 's, not being the components of a tensor, should be hidden from sight in a notation which shows only covariant and absolute derivatives, and the curvature tensor. The formulae (22) and (23) are of fundamental importance.

The operation of absolute differentiation can be applied with respect to each of the parameters for vector or tensor fields defined over a subspace of two or three dimensions. Consider a 2-space with the parametric equations

$$x^i = x^i(u, v). \quad (24)$$

On it we have the two vector fields

$$U^i = \frac{\partial x^i}{\partial u}, \quad V^i = \frac{\partial x^i}{\partial v}. \quad (25)$$

If we take absolute derivatives of these vector fields with respect to v and u respectively, we find by (16) that

$$\frac{\delta U^i}{\delta v} = \frac{\delta V^i}{\delta u}. \quad (26)$$

This result may be seen immediately by recognizing that it is a tensor equation, and that it is true for a coordinate system which makes the Christoffel symbols vanish at the point in question.

Eq. (26) might suggest that the operators $\delta/\delta u$ and $\delta/\delta v$ commute; but that is not true in general (cf. § 5).

The theory of geodesics (in particular, null geodesics) is familiar in tensor calculus. However, with a view to applying the ideas to the world-function in Chapter II, it will be convenient to set up the theory afresh according to the plan adopted by MØLLER [1952, p. 228].

Let C_0 and C_1 be two curves (Fig. 1), and let them be joined by a single infinity of curves such as A_0A_1 and B_0B_1 . The family of joining

curves forms a 2-space with equations $x^i = x^i(u, v)$, where u is a parameter running between fixed end-values (u_0 on C_0 and u_1 on C_1) and v a parameter constant on each of the joining curves. Consider the

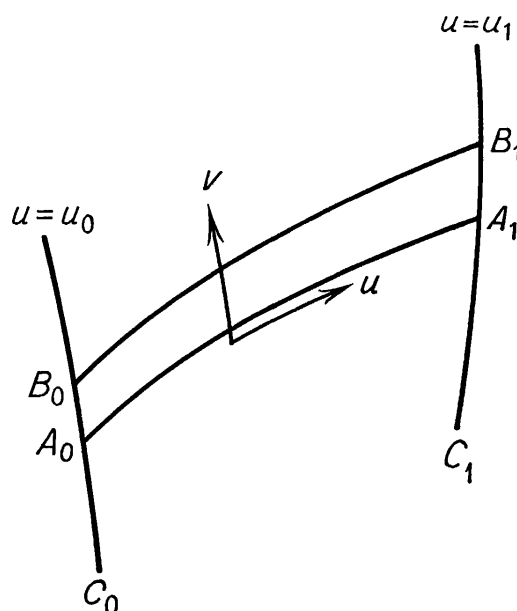


Fig. 1 – The variational problem for geodesics with end-points varied

integral

$$I(v) = \frac{1}{2}(u_1 - u_0) \int_{u_0}^{u_1} g_{ij} \frac{\partial x^i}{\partial u} \frac{\partial x^j}{\partial u} du, \quad (27)$$

taken along any one of the curves $v = \text{const.}$; or, in the notation of (25),

$$I(v) = \frac{1}{2}(u_1 - u_0) \int_{u_0}^{u_1} g_{ij} U^i U^j du. \quad (28)$$

Then, with use of (26), we obtain

$$\begin{aligned} \frac{dI}{dv} &= (u_1 - u_0) \int_{u_0}^{u_1} g_{ij} U^i \frac{\delta V^j}{\delta u} du \\ &= (u_1 - u_0) \int_{u_0}^{u_1} \frac{\partial}{\partial u} (g_{ij} U^i V^j) du - (u_1 - u_0) \int_{u_0}^{u_1} g_{ij} \frac{\delta U^i}{\delta u} V^j du \\ &= (u_1 - u_0) [g_{ij} U^i V^j]_{u_0}^{u_1} - (u_1 - u_0) \int_{u_0}^{u_1} g_{ij} \frac{\delta U^i}{\delta u} V^j du. \end{aligned} \quad (29)$$

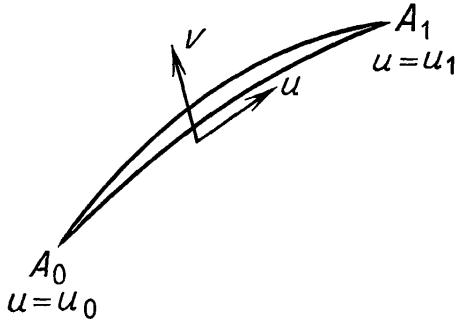


Fig. 2 – The variational problem for geodesics with end-points fixed

We now take the particular case where the curves C_0, C_1 collapse into the points A_0, A_1 (Fig. 2), so that we have a family of curves with fixed end-points. Now $V^j = 0$ at the end-points, and (29) becomes

$$\frac{dI}{dv} = -(u_1 - u_0) \int_{u_0}^{u_1} g_{ij} \frac{\delta U^i}{\delta u} V^j du. \quad (30)$$

We define a *geodesic* to be a curve which gives a stationary value to I for variations which leave the end-points fixed. This demands $dI/dv = 0$ for V^j arbitrary except at the end-points, and so a geodesic satisfies

$$\frac{\delta U^i}{\delta u} = \frac{\delta}{\delta u} \frac{dx^i}{du} = \frac{d^2 x^i}{du^2} + \Gamma_{jk}^i \frac{dx^j}{du} \frac{dx^k}{du} = 0. \quad (31)$$

These equations possess the first integral

$$g_{ij} \frac{dx^i}{du} \frac{dx^j}{du} = \varepsilon k^2, \text{ or } ds = k du, \quad (32)$$

where k is a constant and ε the indicator for a timelike or spacelike geodesic; the geodesic is a *null geodesic* if $k = 0$. We shall, in general, use the word 'geodesic' to include 'null geodesic'.

Every geodesic has a class of *special parameters* for which the equations are as in (31), and the transformation from one special parameter to another is linear:

$$u' = au + b. \quad (33)$$

For other transformations of the parameter, a term proportional to dx^i/du appears on the right hand side.

If the geodesic is not null, we can find special parameters for which $k = 1$, and it is clear that for such a parameter we have $du = ds$. Thus the equation (31) may be written

$$\frac{\delta}{\delta s} \frac{dx^i}{ds} = \frac{d^2 x^i}{ds^2} + \Gamma_{jk}^i \frac{dx^j}{ds} \frac{dx^k}{ds} = 0. \quad (34)$$

In applying the variational principle $\delta I = 0$ for fixed end-points, we have found not only a class of curves (geodesics), but also a class of special parameters on them. In order to interpret the integral I , we note that for any non-null curve $x^i = x^i(u)$ with $u_0 \leq u \leq u_1$ it is possible to pick out a particular parameter, say u' , defined by

$$u' - u_0 = (u_1 - u_0)s/L, \quad (35)$$

where s is the measure of the current point from the point where $u = u_0$, and L is the value of s for $u = u_1$. For such a parameter, the integral I of (27) is easily seen to be

$$I(v) = \frac{1}{2}\varepsilon L^2. \quad (36)$$

This holds even for null curves, since then it reads $0 = 0$.

Thus the variational principle which we have used implies

$$\delta(L^2) = 0, \quad (37)$$

and this implies the more usual variational equation

$$\delta L = \delta f ds = 0, \quad (38)$$

provided $L \neq 0$; the advantage of the method adopted above is that null geodesics fit in without any special treatment, and perhaps it would be well to relegate (38) to the history of science.

What has been done above is closely connected with the theory of the world-function, to be treated in Chapter II. Here we shall only push the argument one stage further. Let us go back to Fig. 1 and equation (29). Let the joining curves in Fig. 1 be geodesics, and let u be a special parameter on each of them, running between the fixed end-values u_0, u_1 . Then (29) reduces to

$$\frac{dI}{dv} = (u_1 - u_0)[g_{ij}U^iV^j]_{u_0}^{u_1}, \quad (39)$$

or, with the variational δ ,

$$\delta I = (u_1 - u_0)[g_{ij}U^i\delta x^j]_{u_0}^{u_1}. \quad (40)$$

Now the integral I is a function of the coordinates of A_0 (say $x^{i'}$) and the coordinates of A_1 (say x^i), and (40) gives for the derivatives of this function

$$\frac{\partial I}{\partial x^i} = (u_1 - u_0) \left(g_{ij} \frac{dx^j}{du} \right)_{A_1}, \quad \frac{\partial I}{\partial x^{i'}} = - (u_1 - u_0) \left(g_{ij} \frac{dx^j}{du} \right)_{A_0}. \quad (41)$$

These formulae hold even for a null geodesic. The right hand sides are invariant under transformation of the special parameter. If the geodesic is not null, we have the simpler formulae

$$\frac{\partial I}{\partial x^i} = L\lambda_i, \quad \frac{\partial I}{\partial x^{i'}} = -L\lambda_{i'}, \quad (42)$$

where $\lambda^{i'}$ and λ^i are the unit tangent vectors to the geodesic at A_0 and A_1 respectively, both of these vectors pointing in the sense A_0A_1 ; L is the finite measure A_0A_1 .

§ 3. ORTHONORMAL TETRADS AND FRENET-SERRET FORMULAE

Four mutually orthogonal unit vectors are said to form an *orthonormal tetrad* (briefly OT). The vectors of an OT may be denoted by $\lambda_{(a)}^i$, where i is the contravariant tensor index and (a) a label distinguishing the particular vector; the covariant components of the same OT are

$$\lambda_{(a)i} = g_{ij}\lambda_{(a)}^j. \quad (43)$$

Three of the vectors of an OT are necessarily spacelike and one is timelike. We shall always so label the vectors that $\lambda_{(4)}^i$ is timelike. Then the conditions of orthonormality may be written

$$\lambda_{(a)}^i \lambda_{(b)i} = \eta_{(ab)}, \quad (44)$$

where

$$\eta_{(ab)} = \eta^{(ab)} = \text{diag}(1, 1, 1, -1), \quad (45)$$

a diagonal invariant matrix with the elements indicated; it satisfies

$$\eta^{(ab)} \eta_{(ac)} = \delta_c^b, \quad (46)$$

and is therefore, in matrix language, a square root of unity.

To secure simplicity later, we introduce a complication. The labels on the vectors have no tensorial meaning, but nevertheless we shall raise and lower them by means of the η -matrix. Thus we define

$$\lambda^{(a)i} = \eta^{(ab)} \lambda_{(b)}^i, \quad \lambda_i^{(a)} = \eta^{(ab)} \lambda_{(b)i}, \quad (47)$$

and deduce, by (46)

$$\lambda_{(a)}^i = \eta_{(ab)} \lambda^{(b)i}, \quad \lambda_{(a)i} = \eta_{(ab)} \lambda_i^{(b)}. \quad (48)$$

Now (44) may be written more neatly as

$$\lambda_{(a)}^i \lambda_i^{(b)} = \delta_a^b, \quad (49)$$

and it is an algebraic consequence of this that

$$\lambda_{(a)}^i \lambda_j^{(a)} = \delta_j^i. \quad (50)$$

The two tetrads $\lambda_{(a)}^i$ and $\lambda^{(a)i}$ are closely connected: their spacelike vectors are the same and their timelike vectors are opposed to one another.

If, at a point in space-time, we are given two OT, $\lambda_{(a)}^i$ and $\mu_{(a)}^i$, they are connected by a Lorentz transformation. To discuss this transformation, we introduce the invariant *Lorentz matrix*

$$L_{(b)}^{(a)} = \lambda_{(b)}^{(a)} \mu_{(a)}^i, \quad (51)$$

so that, in matrix notation, $\mathbf{L} = \mathbf{1}$ if the two OT coincide. Multiplication of (51) in turn by $\mu_j^{(b)}$ and $\lambda_{(a)}^j$ gives

$$\lambda_j^{(a)} = L_{(b)}^{(a)} \mu_j^{(b)}, \quad \mu_{(b)}^j = L_{(b)}^{(a)} \lambda_{(a)}^j. \quad (52)$$

These are equivalent expressions for the Lorentz transformation. We need a name for the indices in parentheses (the labels); we shall call

them *Lorentz indices* to distinguish them from the ordinary tensorial indices.

Multiplying the second of (52) by $\mu_{(c)j}$ and using (44), we get

$$\eta_{(bc)} = L_{(b)}^{(a)} \lambda_{(a)}^j \mu_{(c)j} = L_{(b)}^{(a)} \eta_{(ad)} \lambda_j^{(d)} \mu_{(c)}^j = L_{(b)}^{(a)} \eta_{(ad)} L_{(c)}^{(d)}. \quad (53a)$$

If we regard $L_{(b)}^{(a)}$ as a 4×4 matrix, with the upper index denoting row and the lower index column, we have in matrix notation

$$\tilde{\mathbf{L}} \boldsymbol{\eta} \mathbf{L} = \boldsymbol{\eta}, \quad \mathbf{L} \boldsymbol{\eta} \tilde{\mathbf{L}} = \boldsymbol{\eta}, \quad (53b)$$

the second equation following from the first because $\boldsymbol{\eta}^2 = \mathbf{1}$.

Any vector or any tensor may be resolved into components along an OT $\lambda_{(a)}^i$. These components are invariants in the tensorial sense (i.e. with respect to coordinate transformation), but they depend on the choice of the OT, and are either contravariant or covariant with respect to Lorentz transformations of the OT. We have the following typical formulae:

$$\begin{aligned} V_{(a)} &= V_i \lambda_{(a)}^i, & V^{(a)} &= V^i \lambda_i^{(a)}, \\ V_i &= V_{(a)} \lambda_i^{(a)}, & V^i &= V^{(a)} \lambda_{(a)}^i, \\ T_{(ab)} &= T_{ij} \lambda_{(a)}^i \lambda_{(b)}^j, & T_{ij} &= T_{(ab)} \lambda_i^{(a)} \lambda_j^{(b)}, \\ R_{(abcd)} &= R_{ijkl} \lambda_{(a)}^i \lambda_{(b)}^j \lambda_{(c)}^k \lambda_{(d)}^l, \\ R_{ijkl} &= R_{(abcd)} \lambda_i^{(a)} \lambda_j^{(b)} \lambda_k^{(c)} \lambda_l^{(d)}, \\ R^{ijkl} &= R^{(abcd)} \lambda_{(a)}^i \lambda_{(b)}^j \lambda_{(c)}^k \lambda_{(d)}^l, \\ V_{(a)} &= \eta_{(ab)} V^{(b)}, & V^{(a)} &= \eta^{(ab)} V_{(b)}, \\ T_{(ab)} &= \eta_{(ac)} \eta_{(bd)} T^{(cd)}, & T^{(ab)} &= \eta^{(ac)} \eta^{(bd)} T_{(cd)}. \end{aligned} \quad (54)$$

One quickly gains confidence in the manipulation of the Lorentz indices, and their interplay with tensorial indices. The ‘up-and-down’ rule prevails throughout.

Associated with each point of a curve Γ in space-time (we shall consider only a timelike curve) there is a particularly interesting OT. Consider the following equations:

$$\delta A^i / \delta s = b B^i, \quad (55a)$$

$$\delta B^i / \delta s = c C^i + b A^i, \quad (55b)$$

$$\delta C^i / \delta s = d D^i - c B^i, \quad (55c)$$

$$\delta D^i / \delta s = -d C^i, \quad (55d)$$

together with

$$A^i A_i = -1, \quad B^i B_i = C^i C_i = D^i D_i = 1. \quad (55e)$$

The coefficients b, c, d are non-negative¹ scalars. Let

$$A^i = dx^i/ds, \quad (56)$$

the unit tangent to Γ ; this is consistent with (55e). Then (55a) and (55e) define B^i and b , (55b) and (55e) define C^i and c , and (55c) and (55e) define D^i and d . By (55e) the four vector are unit vectors. We shall show that they form an OT by establishing their orthogonality. Finally we shall verify (55d).

The argument proceeds as follows. By (55e) we have

$$A_i \delta A^i / \delta s = 0, \quad (57)$$

and so (55a) implies

$$A_i B^i = 0; \quad (58)$$

thus B^i is orthogonal to A^i . To show that C^i is orthogonal to A^i and B^i , we form from (55b) the products

$$c A_i C^i = A_i \delta B^i / \delta s + b, \quad (59)$$

$$c B_i C^i = B_i \delta B^i / \delta s. \quad (60)$$

Differentiation of (58), with (55a), gives

$$A_i \delta B^i / \delta s = -B_i \delta A^i / \delta s = -b, \quad (61)$$

and so, by (59), we have $A_i C^i = 0$. By (55e), (60) gives $B_i C^i = 0$. Thus C^i is orthogonal to A^i and B^i . The proof that D^i is orthogonal to A^i, B^i and C^i follows the same lines.

To establish (55d), we note that any vector can be resolved along the tetrad (A, B, C, D) , and so we can write

$$\delta D^i / \delta s = \alpha A^i + \beta B^i + \gamma C^i + \delta D^i. \quad (62)$$

Multiplying this equation in turn by A_i, B_i, C_i, D_i , and using (55a-c) and the orthogonalities already established, we find that

$$\alpha = \beta = \delta = 0, \quad \gamma = -d, \quad (63)$$

and so (55d) is verified.

The formulae (55) are the *Frenet-Serret formulae*; B^i, C^i, D^i are the

¹ If zeros of b, c, d occur on Γ , it is better to allow these scalars to take both positive and negative values in order to secure continuity of the vectors B^i, C^i, D^i .

first, second and third normals to I , respectively, and b, c, d are the first, second and third curvatures.

The simplest of all timelike curves is a geodesic. For it we have $b = c = d = 0$. The next simplest is the timelike *circle*¹, defined by

$$b = \text{const.}, \quad c = d = 0; \quad (64)$$

for it the Frenet-Serret formulae reduce to

$$\delta A^i / \delta s = b B^i, \quad \delta B^i / \delta s = b A^i, \quad \delta C^i / \delta s = 0, \quad \delta D^i / \delta s = 0. \quad (65)$$

Then we have the timelike *helix* with

$$b = \text{const.}, \quad c = \text{const.}, \quad d = 0, \quad (66)$$

$$\delta A^i / \delta s = b B^i, \quad \delta B^i / \delta s = c C^i + b A^i, \quad \delta C^i / \delta s = -c B^i, \quad \delta D^i / \delta s = 0. \quad (67)$$

§ 4. PARALLEL TRANSPORT AND FERMI-WALKER TRANSPORT

Consider a curve I with equations $x^i = x^i(u)$ and a vector field V^i defined along I . The vector is said to undergo *parallel transport* (LEVI-CIVITÀ [1917d]) along I if its absolute derivative vanishes:

$$\frac{\delta V^i}{\delta u} = 0. \quad (68)$$

This definition is valid even if I is a null curve. Further, (68) is invariant in form under transformation of the parameter u .

It is obvious that under parallel transport the magnitude of a vector is unchanged, and the scalar product of two vectors is unchanged:

$$\left. \begin{aligned} \frac{\delta U^i}{\delta u} = 0 \\ \frac{\delta V^i}{\delta u} = 0 \end{aligned} \right\} \Rightarrow \left\{ \begin{aligned} \frac{d}{du} (U_i U^i) = 0, \quad \frac{d}{du} (V_i V^i) = 0, \\ \frac{d}{du} (U_i V^i) = 0. \end{aligned} \right. \quad (69)$$

The eq. (68) defines V^i along I if V^i is given at any one point on I . It is clear from (69) that an OT $\lambda_{(a)}^i$ [cf. § 3] remains an OT under parallel transport; and if a vector V^i also undergoes parallel transport, then its components $V_{(a)}$ on the OT, as in (54), remain constant.

The unit tangent vector to a geodesic undergoes parallel transport because, as in (34),

$$\frac{\delta}{\delta s} \frac{dx^i}{ds} = 0. \quad (70)$$

¹ This circle is not a closed curve. One might prefer to call it *hyperbola of constant curvature*.

In the case of a null geodesic, the tangent vector dx^i/du undergoes parallel transport, provided u is a special parameter; for then, as in (31),

$$\frac{\delta}{\delta u} \frac{dx^i}{du} = 0. \quad (71)$$

Consider now a timelike curve I with equations $x^i = x^i(s)$. For the unit tangent vector we write $A^i = dx^i/ds$. We define *Fermi-Walker transport*¹ (briefly, F-W transport) of a vector F^i along I by the equation

$$\frac{\delta F^i}{\delta s} = bF_j(A^iB^j - A^jB^i), \quad (72)$$

where B^i and b are the first normal and curvature of I , as in (55a). As in the case of parallel transport, this equation defines F^i along I if F^i is given at any one point of I .

An important feature of F-W transport is that the unit tangent vector A^i itself automatically undergoes F-W transport; for it is easy to verify that

$$\frac{\delta A^i}{\delta s} = bA_j(A^iB^j - A^jB^i), \quad (73)$$

by virtue of (55a), (55e) and (58).

F-W transport resembles parallel transport in the conservation of magnitude and scalar product. For, if U_i and V_i both undergo F-W transport, then

$$\frac{d}{ds} (V_i V^i) = 2V_i \frac{\delta V^i}{\delta s} = 2bV_i V_j (A^iB^j - A^jB^i) = 0, \quad (74)$$

$$\begin{aligned} \frac{d}{ds} (U_i V^i) &= U_i \frac{\delta V^i}{\delta s} + V_i \frac{\delta U^i}{\delta s} \\ &= b(U_i V_j + V_i U_j)(A^iB^j - A^jB^i) = 0. \end{aligned} \quad (75)$$

It is clear that, as in parallel transport, F-W transport conserves an OT and the components of a vector on the OT.

Since it is defined by a simpler equation, parallel transport is more fundamental mathematically than F-W transport, but F-W transport is more important in some physical situations. The reason for this is

¹ FERMI [1922]; WALKER [1932].

shown in Fig. 3. If we take an OT on Γ , with the fourth member tangent to Γ at some point (so that $\lambda_{(4)}^i = A^i$), then, under parallel transport, $\lambda_{(4)}^i$ wanders away from A^i (unless Γ happens to be a geodesic). But under F-W transport $\lambda_{(4)}^i$ remains tangent to Γ . Thus F-W transport not only provides us with an OT along Γ ; it also provides an orthonormal *triad* orthogonal to Γ . This forms a 'spatial frame of reference' for an observer whose history in space-time is Γ and, as we shall see later on, it is this frame of reference which appears to give us the correct relativistic generalization of the Newtonian concept of a 'non-rotating frame'.

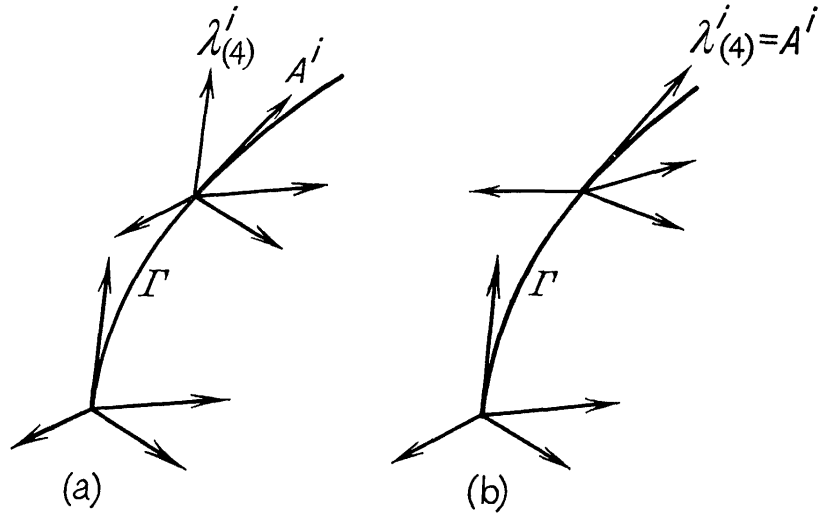


Fig. 3 – (a) Parallel transport. (b) Fermi-Walker transport

If Γ is a geodesic, parallel transport and F-W transport coincide [put $b = 0$ in (72)], provided Γ is not a null geodesic. Since F-W transport involves s , it is undefined along any null line.

To complete this discussion of F-W transport, let us consider an OT $\lambda_{(a)}^i$ undergoing F-W transport along Γ , with

$$\lambda_{(4)}^i = A^i, \quad (76)$$

the unit tangent to Γ . Each of the four vectors $\delta\lambda_{(a)}^i/\delta s$ can be resolved on the tetrad, so that we may write

$$\frac{\delta\lambda_{(a)}^i}{\delta s} = Q_{(ab)}\eta^{(bc)}\lambda_{(c)}^i, \quad (77)$$

the η -factor (45) being inserted for notational convenience. Multiply by $\lambda_{(d)i}$ and use (44): this gives

$$Q_{(ad)} = \lambda_{(d)i} \frac{\delta\lambda_{(a)}^i}{\delta s}, \quad (78)$$

because

$$\eta^{(bc)}\eta_{(cd)} = \delta_d^b. \quad (79)$$

But, by (72),

$$\frac{\delta\lambda_{(a)}^i}{\delta s} = b\lambda_{(a)j}(\lambda_{(4)}^i B^j - \lambda_{(4)}^j B^i), \quad (80)$$

and so

$$Q_{(ad)} = b(B_{(a)}\eta_{(d4)} - B_{(d)}\eta_{(a4)}), \quad (81)$$

where

$$B_{(a)} = B_i \lambda_{(a)}^i, \quad (82)$$

the invariant components of the first normal vector on the tetrad [cf. (54)]. The invariant matrix \mathbf{Q} is important because it sums up the behaviour of the OT under F-W transport. It is skew-symmetric, and all its elements vanish identically except the following:

$$Q_{(4\alpha)} = -Q_{(\alpha 4)} = bB_{(\alpha)}, \quad (83)$$

the Greek suffix running 1, 2, 3.

If F-W transport is applied in particular to a vector F^i which is orthogonal to the tangent A^i at some point on Γ , it of course remains orthogonal to A^i , and the formula (72) then simplifies to

$$\frac{\delta F^i}{\delta s} = bA^i F_j B^j. \quad (84)$$

This is the transport law originally given by FERMI [1922]. We shall refer to this as *Fermi transport*, but we shall use it only for vectors which are orthogonal to A^i .

The rotation of the orthonormal triad of normals (B^i, C^i, D^i) of § 3, relative to an orthonormal triad of Fermi vectors $\lambda_{(\alpha)}^i$, is discussed in III-§ 9.

§ 5. THE TENSORS OF RIEMANN, RICCI AND EINSTEIN

The *Riemann tensor* (or *curvature tensor*) can be expressed in several equivalent forms as follows:

$$\begin{aligned} R_{ijkl} = & \frac{1}{2}(g_{im,jk} + g_{jk,im} - g_{ik,jm} - g_{jm,ik}) \\ & + g^{ab}([im,a][jk,b] - [ik,a][jm,b]), \end{aligned} \quad (85)$$

$$R_{ijklm} = \frac{1}{2}(g_{im,jk} + g_{jk,im} - g_{ik,jm} - g_{jm,ik}) + g_{ab}(\Gamma_{im}^a \Gamma_{jk}^b - \Gamma_{ik}^a \Gamma_{jm}^b), \quad (86)$$

$$R_{ijklm} = [jm,i]_{,k} - [jk,i]_{,m} + \Gamma_{jk}^a [im,a] - \Gamma_{jm}^a [ik,a], \quad (87)$$

$$R_{.jkm}^i = \Gamma_{jm,k}^i - \Gamma_{jk,m}^i + \Gamma_{jm}^a \Gamma_{ak}^i - \Gamma_{jk}^a \Gamma_{am}^i. \quad (88)$$

The Riemann tensor satisfies the following symmetry equations:

$$R_{ijklm} = -R_{jiklm} = -R_{ijmkl} = R_{kmlji}, \quad (89)$$

$$R_{iabc} + R_{ibca} + R_{icab} = 0. \quad (90)$$

There are 20 independent components, and they are most compactly exhibited in a notation which correlates ordered pairs of numbers from the range 1, 2, 3, 4 with the numbers 1, 2, . . . 6 according to the scheme

$$23 \leftrightarrow 1, \quad 31 \leftrightarrow 2, \quad 12 \leftrightarrow 3, \quad 14 \leftrightarrow 4, \quad 24 \leftrightarrow 5, \quad 34 \leftrightarrow 6. \quad (91)$$

Capital letters having the range 1, 2, . . . 6, all non-zero covariant components of the Riemann tensor are comprised in the symmetric 6×6 matrix \bar{R}_{AB} , where, for example,

$$R_{2331} = R_{3123} = \bar{R}_{12} = \bar{R}_{21}. \quad R_{2314} = R_{1423} = \bar{R}_{14} = \bar{R}_{41}. \quad (92)$$

This symmetric matrix contains 21 elements when allowance is made for its symmetry. The cyclic identity (90) imposes the equation

$$R_{2314} + R_{3124} + R_{1234} = \bar{R}_{14} + \bar{R}_{25} + \bar{R}_{36} = 0, \quad (93)$$

reducing the number from 21 to 20.

For covariant differentiation and for absolute differentiation on a 2-space $x^i = x^i(u, v)$ the commutation rules are as follows (we write $U^i = \partial x^i / \partial u$, $V^i = \partial x^i / \partial v$):

$$T_{i|jk} - T_{i|kj} = R_{.ijk}^a T_a, \quad (94)$$

$$\frac{\delta^2 T^i}{\delta u \delta v} - \frac{\delta^2 T^i}{\delta v \delta u} = R_{.abc}^i T^a U^b V^c, \quad (95)$$

$$T_{ij|km} - T_{ij|m k} = R_{.ikm}^a T_{aj} + R_{.jkm}^a T_{ia}, \quad (96)$$

$$\frac{\delta^2 T^{ij}}{\delta u \delta v} - \frac{\delta^2 T^{ij}}{\delta v \delta u} = R_{.abc}^i T^{aj} U^b V^c + R_{.abc}^j T^{ia} U^b V^c. \quad (97)$$

Similar commutation rules hold for tensors of higher orders. The rules are most easily verified by using coordinates which make $\Gamma_{jk}^i = 0$ at the point under consideration.

The Riemann tensor satisfies the *Bianchi identities*:

$$R_{ijab|c} + R_{ijbc|a} + R_{ijca|b} = 0. \quad (98)$$

The *Riemannian curvature* associated with a pair of vectors ξ^i, η^i is the invariant

$$K = \frac{R_{ijkl}\xi^i\eta^j\xi^k\eta^l}{g_{abcd}\xi^a\eta^b\xi^c\eta^d} \quad (99)$$

where

$$g_{abcd} = g_{ac}g_{bd} - g_{ad}g_{bc}. \quad (100)$$

This last tensor has the same symmetry as the Riemann tensor.

For a space of constant curvature K , we have

$$R_{ijkl} = Kg_{ijkl}. \quad (101)$$

Space-time is flat if, and only if, $R_{ijkl} = 0$. In flat space-time there exist coordinates such that

$$g_{ij} = \eta_{ij} = \text{diag}(1, 1, 1, -1), \quad (102)$$

$$\begin{aligned} ds^2 &= \varepsilon \eta_{ij} dx^i dx^j \\ &= \varepsilon [(dx^1)^2 + (dx^2)^2 + (dx^3)^2 - (dx^4)^2]. \end{aligned} \quad (103)$$

This reduction can, of course, always be effected at a single point.

The *Ricci tensor* R_{ij} is defined by ¹

$$R_{ij} = R^k_{.ijk} = g^{km} R_{kijm} = R_{ji}. \quad (104)$$

Explicitly we have

$$R_{ij} = \Gamma^a_{ai,j} - \Gamma^a_{ij,a} + \Gamma^a_{bi}\Gamma^b_{aj} - \Gamma^a_{ij}\Gamma^b_{ab}, \quad (105)$$

$$R_{ij} = \frac{1}{2}[\log(-g)]_{,ij} - \Gamma^a_{ij,a} - \frac{1}{2}\Gamma^a_{ij}[\log(-g)]_{,a} + \Gamma^a_{bi}\Gamma^b_{aj}. \quad (106)$$

The *curvature invariant* is

$$R = g^{ij}R_{ij} = R^i_i, \quad (107)$$

and the *Einstein tensor* is

$$G_{ij} = R_{ij} - \frac{1}{2}g_{ij}R = G_{ji}, \quad (108)$$

or, in mixed form,

$$G^i_j = R^i_j - \frac{1}{2}\delta^i_j R. \quad (109)$$

¹ Some authors reverse the sign.

By virtue of the Bianchi identities (98), the Einstein tensor satisfies the four *conservation equations* (or identities)

$$G^a_{i|a} = 0. \quad (110)$$

These equations are important physically in connection with the conservation of momentum and energy. They may also be written

$$G^{ia}_{|a} = 0. \quad (111)$$

If we multiply (110) by $(-g)^{\frac{1}{2}}$, the equation may be changed into the following alternative forms: ¹

$$[(-g)^{\frac{1}{2}}G^a_i]_{,a} - \frac{1}{2}(-g)^{\frac{1}{2}}G^{ab}g_{ab,i} = 0, \quad (112)$$

$$[(-g)^{\frac{1}{2}}G^a_i]_{,a} + \frac{1}{2}(-g)^{\frac{1}{2}}G_{ab}g^{ab}_{,i} = 0. \quad (113)$$

The numerical permutation symbol ε_{ijklm} is defined by the following properties: (a) it vanishes if two suffixes are the same, (b) $\varepsilon_{ijklm} = 1$, (c) it changes sign when two suffixes are interchanged. The *permutation tensor* ² is, in contravariant and covariant forms,

$$\eta^{ijklm} = (-g)^{-\frac{1}{2}}\varepsilon_{ijklm}, \quad \eta_{ijklm} = -(-g)^{\frac{1}{2}}\varepsilon_{ijklm}. \quad (114)$$

The *double dual* ³ of the Riemann tensor is

$$\tilde{R}^{ijklm} = \frac{1}{4}\eta^{ijab}R_{abcd}\eta^{cdkm}, \quad (115)$$

or equivalently in mixed form

$$\begin{aligned} \tilde{R}^{ij}_{\cdot\cdot km} &= \frac{1}{4}\eta^{ijab}R_{ab}^{\cdot cd}\eta_{cdkm} \\ &= -\frac{1}{4}\varepsilon_{ijab}R^{\cdot cd}_{\cdot ab}\varepsilon_{cdkm}. \end{aligned} \quad (116)$$

Thus, for example,

$$\tilde{R}^{23}_{\cdot\cdot 23} = -R^{14}_{\cdot\cdot 14}, \quad \tilde{R}^{23}_{\cdot\cdot 31} = -R^{24}_{\cdot\cdot 14}, \quad \tilde{R}^{12}_{\cdot\cdot 34} = -R^{12}_{\cdot\cdot 34}. \quad (117)$$

Also, by (115),

$$g\tilde{R}^{2314} = -R_{1423}, \quad g\tilde{R}^{3124} = -R_{2431}, \quad g\tilde{R}^{1234} = -R_{3412}. \quad (118)$$

¹ Cf. MØLLER [1952], pp. 337, 338.

² The notation here used for these *oriented tensors* differs from that used in SYNGE and SCHILD [1956], p. 249. In (114) the second η -symbol is obtained from the first by lowering the superscripts in the usual way.

³ One often uses a single star for the dual and a double star for the double dual; cf. X-§ 3.

The double dual tensor satisfies the symmetry equations

$$\tilde{R}^{ijkm} = -\tilde{R}^{jikm} = -\tilde{R}^{ijmk} = \tilde{R}^{kmij}, \quad (119)$$

$$\tilde{R}^{iabc} + \tilde{R}^{ibca} + \tilde{R}^{icab} = 0. \quad (120)$$

Except for the raising of the subscripts, these are of the same form as the symmetry equations (89), (90) for the Riemann tensor. Eq. (119) are obvious, and (119) imply (120) unless $(iabc)$ are distinct, i.e. a permutation of (1234). Thus (120) follows from (118) and (93).

The double dual tensor is connected in an interesting way with the Einstein tensor. Putting $m = i$ in (116), we get

$$\tilde{R}^{ij}_{..ki} = \frac{1}{4} \delta_{kcd}^{jab} R^{cd}_{..ab}, \quad (121)$$

where

$$\delta_{kcd}^{jab} = \varepsilon_{ijab} \varepsilon_{ikcd}. \quad (122)$$

This generalized Kronecker delta is a tensor, and obeys the following rules: (i) it vanishes unless (jab) are distinct numbers and (kcd) is a permutation of (jab) , and (ii) it is $+1$ or -1 according as that permutation is even or odd. In fact

$$\delta_{kcd}^{jab} = \begin{vmatrix} \delta_k^j & \delta_c^j & \delta_d^j \\ \delta_k^a & \delta_c^a & \delta_d^a \\ \delta_k^b & \delta_c^b & \delta_d^b \end{vmatrix}. \quad (123)$$

Substituting this in (121), we get, after a simple reduction ¹,

$$\tilde{R}^{ij}_{..ki} = G^j_k. \quad (124)$$

For the symmetrized Riemann tensor, see II-§ 2.

§ 6. THE DEVIATION OF GEODESICS

Consider a single infinity of curves $\Gamma(v)$ with equations $x^i = x^i(u, v)$, where $v = \text{const.}$ along each curve (Fig. 4). They form a 2-space. Let us write

$$\partial x^i / \partial u = U^i, \quad \partial x^i / \partial v = V^i; \quad (125)$$

then, by (26),

$$\delta U^i / \delta v = \delta V^i / \delta u. \quad (126)$$

¹ Cf. LANCZOS [1938].

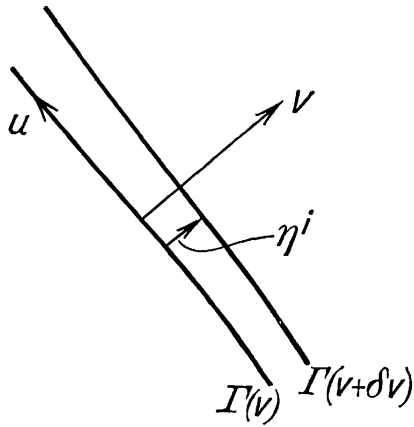


Fig. 4 – Deviation of $\Gamma(v + \delta v)$ from $\Gamma(v)$

In studying a pair of adjacent curves, $\Gamma(v)$ and $\Gamma(v + \delta v)$, it is sometimes intuitively pleasant to deal with the infinitesimal *deviation vector* η^i ,

$$\eta^i = V^i \delta v; \quad (127)$$

but since δv is merely an infinitesimal constant, η^i and V^i are essentially equivalent to one another, and we avoid those mental confusions associated with infinitesimals by using the finite vector V^i .

To find out how $\Gamma(v + \delta v)$ deviates from $\Gamma(v)$, we write down the following equations, using (126) and (95):

$$\frac{\delta^2 V^i}{\delta u^2} = \frac{\delta}{\delta u} \frac{\delta V^i}{\delta u} = \frac{\delta}{\delta u} \frac{\delta U^i}{\delta v} = \frac{\delta}{\delta v} \frac{\delta U^i}{\delta u} + R^i_{jkm} U^j U^k V^m. \quad (128)$$

So far the curves $\Gamma(v)$ have been general. Henceforth we shall take them to be geodesics (perhaps some or all of them are null geodesics), the parameter u on each of them being a special parameter, so that, by (31),

$$\frac{\delta U^i}{\delta u} = 0. \quad (129)$$

The first term on the right hand side of (128) vanishes, and we have the *equation of geodesic deviation*

$$\frac{\delta^2 V^i}{\delta u^2} + R^i_{jkm} U^j V^k U^m = 0, \quad (130)$$

or, equivalently,

$$\frac{\delta^2 \eta^i}{\delta u^2} + R^i_{jkm} U^j \eta^k U^m = 0. \quad (131)$$

Since u is a special parameter on each of the curves, the correspondence between points on $\Gamma(v)$ and $\Gamma(v + \delta v)$ is not of a general character, because the special parameter on any geodesic can be subjected only to a *linear* transformation (cf. § 2). We have, in fact,

$$\frac{\partial}{\partial u} (U_i V^i) = U_i \frac{\delta V^i}{\delta u} = U_i \frac{\delta U^i}{\delta v} = \frac{1}{2} \frac{\partial}{\partial v} (U_i U^i). \quad (132)$$

Under two circumstances this vanishes: (a) if the curves $\Gamma(v)$ are null geodesics, for then $U_i U^i = 0$, or (b) if the curves are ordinary geodesics and $u = s$ on each of them, for then $U_i U^i = \pm 1$. Under either of these conditions we have, along $\Gamma(v)$,

$$U_i V^i = \text{const.}, \quad \text{or} \quad \eta_i \frac{\partial x^i}{\partial u} = \text{const.} \quad (133)$$

In particular, if \dot{V}^i (or η^i) is orthogonal to $\Gamma(v)$ at any point, it remains orthogonal.

Null geodesics are of great importance in relativity, because nearly all astronomical information comes to us optically, i.e. by photons, and, as we shall see later, the history of a photon is a null geodesic in space-time. In preparation for later physical developments, we shall pursue the geometry of null geodesics a little further here.

Let C_1 and C_2 (Fig. 5) be two timelike curves (not necessarily geodesics, although they might be); they represent an observer and a source of light respectively. Let P_1 be any point on C_1 . The totality of null geodesics drawn through P_1 form a *null cone*, with two sheets, which are called the *past sheet* and the *future sheet* (see Chap. III). We here consider the past sheet only. It is cut by the curve C_2 at some point P_2 , so that we may say that the null cone

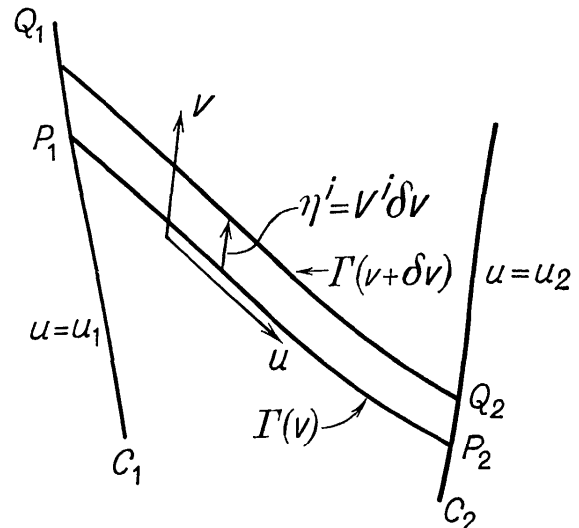


Fig. 5 – Mapping by null geodesics, and their deviation

maps P_1 on P_2 . Thus the whole curve C_1 is mapped pointwise on C_2 ; Fig. 5 shows two of the null geodesics (P_1P_2 , Q_1Q_2) which effect this mapping. The totality of these null geodesics form a 2-space, which is determined once C_1 and C_2 are given.

Let u_1 and u_2 be any two numbers. Since we have the liberty of a linear transformation (and that only) in the choice of special parameters on a null geodesic, there exists on each of the null geodesics considered above a unique special parameter u for which $u = u_1$ on C_1 and $u = u_2$ on C_2 . The parameter u being so defined over the 2-space, we choose a second parameter v which is constant along each of the null geodesics (we might choose $v = s$ on C_1). Now we are back

in the situation dealt with earlier, and we can apply the equation of geodesic deviation to the system of null geodesics represented by P_1P_2 and Q_1Q_2 . The infinitesimal vectors P_1Q_1 and P_2Q_2 represent the infinitesimal deviation vector η^i at P_1 and P_2 respectively. As in (133), we have

$$U_i V^i = \text{function of } v \text{ only.} \quad (134)$$

Let us now return to the deviation equation (130) and discuss its solutions. What follows applies generally to a family of geodesics, whether null or not; the condition (134) may hold or it may not.

In (130) we have four ordinary differential equations satisfied by the four functions $V^i(u)$ along the curve Γ . Let $\lambda_{(a)}^i$ be an orthonormal tetrad (OT, cf. § 3) chosen arbitrarily at some point of Γ , and defined along Γ by parallel transport, so that we have

$$\frac{\delta \lambda_{(a)}^i}{\delta u} = 0. \quad (135)$$

Multiplying (130) by $\lambda_{(a)i}$, we get

$$D^2 V_{(a)} + R_{ijkm} \lambda_{(a)}^i U^j V^k U^m = 0, \quad (136)$$

where $D = d/du$ and $V_{(a)}$ are invariant components on the OT, so that, as in (54),

$$V_{(a)} = V_i \lambda_{(a)}^i, \quad V^i = V^{(a)} \lambda_{(a)}^i. \quad (137)$$

Let us introduce other invariant components:

$$U_{(a)} = U_i \lambda_{(a)}^i, \quad R_{(abcd)} = R_{ijkl} \lambda_{(a)}^i \lambda_{(b)}^j \lambda_{(c)}^k \lambda_{(d)}^l. \quad (138)$$

We note that, by (129) and (135),

$$U_{(a)} = \text{const.} \quad (139)$$

along Γ . In terms of the invariant components, we can now write (136) in the form

$$D^2 V^{(a)} + K_{(c)}^{(a)} V^{(c)} = 0, \quad (140)$$

where

$$\begin{aligned} K_{(c)}^{(a)} &= R_{(c}^{(a}{}_{bcd)} U^{(b)} U^{(d)} \\ &= \eta^{(ai)} R_{(ibcd)} U^{(b)} U^{(d)}. \end{aligned} \quad (141)$$

We have thus passed from the tensorial deviation equation (130) to the *invariant deviation equation* (140), which it is convenient to

write in matrix notation as

$$D^2\mathbf{V} + \mathbf{K}\mathbf{V} = 0, \quad (142)$$

\mathbf{K} being the 4×4 invariant matrix (141) and \mathbf{V} the column matrix $V^{(a)}$.

The motivation behind what follows is that we seek to solve (142) in the range $u_1 \leq u \leq u_2$ with assigned values of \mathbf{V} at the ends of the range. However, under certain circumstances a solution does not exist, and it is less confusing to study any existent solution, writing

$$\mathbf{V}(u_1) = \mathbf{V}_1, \quad \mathbf{V}(u_2) = \mathbf{V}_2. \quad (143)$$

Let $G(u, u')$ be the Green's function defined by

$$G(u, u') = \begin{cases} k(u - u_1)(u_2 - u') & \text{for } u \leq u', \\ k(u' - u_1)(u_2 - u) & \text{for } u \geq u', \end{cases} \quad (144)$$

where

$$k = (u_2 - u_1)^{-1}. \quad (145)$$

Writing $D = \partial/\partial u$, $D' = \partial/\partial u'$, we have then

$$\begin{aligned} DG &= k(u_2 - u'), & D'G &= -k(u - u_1) & \text{for } u \leq u', \\ DG &= -k(u' - u_1), & D'G &= k(u_2 - u) & \text{for } u \geq u'. \end{aligned} \quad (146)$$

Now multiply (142) by $G(u, u')du$, with u' arbitrary in the range $u_1 \leq u' \leq u_2$, and integrate over this range. Integrating by parts and noting that $G(u, u') = 0$ at the ends of the range, we get

$$\int_{u_1}^{u_2} DGD\mathbf{V}du = \int_{u_1}^{u_2} G\mathbf{K}\mathbf{V}du. \quad (147)$$

But, by (146), DG is constant in each of the two parts into which u' divides the whole range, and so, if we split the range of integration, it comes outside the integral sign. Thus we get

$$k(u_2 - u')(\mathbf{V}' - \mathbf{V}_1) - k(u' - u_1)(\mathbf{V}_2 - \mathbf{V}') = \int_{u_1}^{u_2} G\mathbf{K}\mathbf{V}du, \quad (148)$$

where $\mathbf{V}' = \mathbf{V}(u')$. Hence

$$\mathbf{V}' = k(u_2 - u')\mathbf{V}_1 + k(u' - u_1)\mathbf{V}_2 + \int_{u_1}^{u_2} G\mathbf{K}\mathbf{V}du. \quad (149)$$

This is an integral equation for \mathbf{V} , incorporating the end-values. If

these end-values are consistent with a unique solution, we can obtain that solution by iteration; thus

$$\begin{aligned} \mathbf{V}' &= k(u_2 - u')\mathbf{V}_1 + k(u' - u_1)\mathbf{V}_2 \\ &\quad + k \int_{u_1}^{u_2} G\mathbf{K}[(u_2 - u)\mathbf{V}_1 + (u - u_1)\mathbf{V}_2]du + O_2, \end{aligned} \quad (150)$$

where O_2 stands for terms involving \mathbf{K} in the second and higher degrees. Since, in the relativistic applications, the curvature of space-time (and hence \mathbf{K}) is small, it usually suffices to retain in (150) only the terms shown explicitly.

With regard to those end-values of \mathbf{V} which are consistent with a solution, we note that if (133) holds, as it does in the case of null geodesics, then these end-values must satisfy

$$U_{(a)}(V^{(a)})_{u=u_1} = U_{(a)}(V^{(a)})_{u=u_2}; \quad (151)$$

we recall that $U_{(a)} = \text{const.}$ along I .

To investigate the first derivative of a solution, we differentiate (149) with respect to u' . To carry this out, we split the range of the integral at u' , and differentiate with respect to u' as a limit of integration and also with respect to u' as a parameter in G . However, since $G(u, u')$ is continuous across $u = u'$, we get zero from differentiation with respect to the limits, and so, using the values of $D'G$ from (146), we obtain

$$D'\mathbf{V}' = k(\mathbf{V}_2 - \mathbf{V}_1) - k \int_{u_1}^{u'} (u - u_1)\mathbf{K}\mathbf{V}du + k \int_{u'}^{u_2} (u_2 - u)\mathbf{K}\mathbf{V}du. \quad (152)$$

Thus, in particular,

$$(D\mathbf{V})_{u=u_1} = k(\mathbf{V}_2 - \mathbf{V}_1) + k \int_{u_1}^{u_2} (u_2 - u)\mathbf{K}\mathbf{V}du. \quad (153)$$

If we now substitute in the integral for \mathbf{V} from (150), we get

$$\begin{aligned} (D\mathbf{V})_{u=u_1} &= k(\mathbf{V}_2 - \mathbf{V}_1) + k^2 \int_{u_1}^{u_2} (u_2 - u)\mathbf{K}[(u_2 - u)\mathbf{V}_1 \\ &\quad + (u - u_1)\mathbf{V}_2]du + O_2. \end{aligned} \quad (154)$$

A case of particular physical interest is that in which we are dealing with null geodesics as in Fig. 5, and the curves C_1 and C_2 are far apart, the curvature of space-time being extremely small except fairly near these curves. We idealize this situation, for mathematical simplicity,

by assuming space-time to be *flat* for

$$u_1 \leq \bar{u}_1 \leq u \leq \bar{u}_2 \leq u_2,$$

so that we have a 'cut-off' as shown in Fig. 6. Since $\mathbf{K} = 0$ for $\bar{u}_1 \leq u \leq \bar{u}_2$, (153) gives

$$\begin{aligned} (DV)_{u=u_1} &= k(\mathbf{V}_2 - \mathbf{V}_1) + k \int_{u_1}^{\bar{u}_1} (u_2 - u) \mathbf{K} \mathbf{V} du \\ &\quad + k \int_{\bar{u}_2}^{u_2} (u_2 - u) \mathbf{K} \mathbf{V} du. \end{aligned} \quad (155)$$

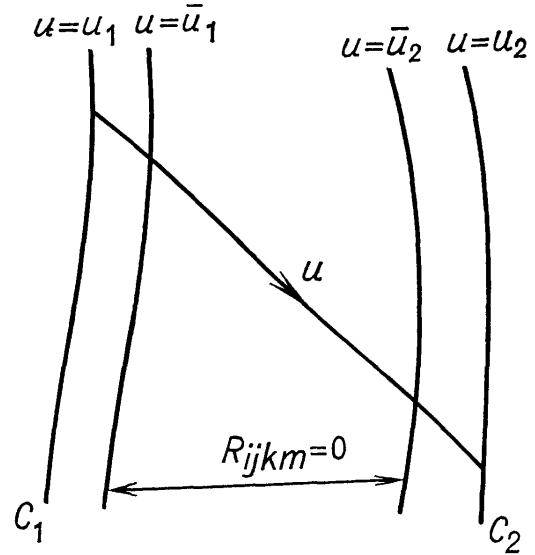


Fig. 6 – Cut-off of curvature

To allow for the curves C_1 and C_2

being 'far apart', we now let $(u_2 - u_1)$ tend to infinity (i.e. $k \rightarrow 0$), at the same time holding $(\bar{u}_1 - u_1)$ and $(u_2 - \bar{u}_2)$ finite. Then the product $k(u_2 - u)$ tends to unity in the first integral and to zero in the second integral, so that we get

$$(DV)_{u=u_1} = \int_{u_1}^{\bar{u}_1} \mathbf{K} \mathbf{V} du. \quad (156)$$

If we now substitute for \mathbf{V} in the integral from (150), we obtain the simple formula

$$(DV)_{u=u_1} = \left(\int_{u_1}^{\bar{u}_1} \mathbf{K} du \right) \mathbf{V}_1 + O_2. \quad (157)$$

This formula expresses the initial value of the derivative of the deviation in terms of the initial value of the deviation and an integral of the Riemann tensor. From the standpoint of pure geometry, the argument may seem a little tedious on account of the artificial character of the assumptions made. But when we come to deal with aberration in Chap. XI, it will be convenient to have the idea of a cut-off already explained.

The deviation of geodesics is discussed again in II-§ 3 with a different notation which uses parallel propagators.

§ 7. HAMILTONIAN THEORY OF RAYS AND WAVES

The Hamiltonian theory of rays and waves is of a very general nature and is best explained in a rather abstract form ¹. Thus, while

¹ For various aspects of Hamiltonian theory, see SYNGE [1954a], [1956b], and a forthcoming article on Classical Dynamics in *Handbuch der Physik*, Vol. 3.

we continue to think of 4-dimensional space-time, for the present we shall not assume the existence of a tensor g_{ij} .

For the moment we take a still wider view and think of an 8-dimensional space V_8 with coordinates (x^i, y_i) , using a subscript for the y 's for reasons which will appear later. The essence of the Hamiltonian approach lies in the assumption of some *Hamiltonian surface* Σ in V_8 , a subspace of seven dimensions for which we write the equation

$$\omega(x, y) = 0. \quad (158)$$

It is important to note that it is the *surface* Σ which is given, and not the *function* $\omega(x, y)$. In elementary geometry, a unit circle may be represented by a great variety of equations, such as

$$\begin{aligned} x^2 + y^2 = 1, \quad (x^2 + y^2)^{\frac{1}{2}} = 1, \quad (x^2 + y^2)^2 = 1, \\ (x^2 + y^2 - 1)^2 = 0, \end{aligned} \quad (159)$$

and in the same way Σ may be represented by a great variety of equations, i.e. by a great variety of functions ω in (158). However, no matter what function ω we select to represent Σ , a displacement $(\delta x^i, \delta y_i)$ lies in Σ if, and only if,

$$\frac{\partial \omega}{\partial x^i} \delta x^i + \frac{\partial \omega}{\partial y_i} \delta y_i = 0; \quad (160)$$

thus, at an assigned point of Σ , the *ratios* of the eight quantities

$$\frac{\partial \omega}{\partial x^i}, \quad \frac{\partial \omega}{\partial y_i} \quad (161)$$

have definite values. In fact, we may write

$$\frac{\partial \omega}{\partial x^i} = \theta \phi_i, \quad \frac{\partial \omega}{\partial y_i} = \theta \psi_i, \quad (162)$$

where the eight quantities ϕ_i, ψ_i are given once for all over Σ and θ is an arbitrary function of position on Σ .

Consider any curve Γ in Σ and the integral

$$I = \int y_i dx^i \quad (163)$$

taken along Γ . We seek the extremals of I ($\delta I = 0$), with the varied curve also lying on Σ ; to allow for this side-condition, we replace the

integral I by

$$J = \int (y_i dx^i - \lambda \omega du), \quad (164)$$

where u is a parameter running between the same end-values for all the curves under consideration and $\lambda(u)$ is a Lagrange multiplier. Applying a variation and integrating by parts, we get

$$\begin{aligned} \delta J = [y_i \delta x^i] + \int \left(\delta y_i dx^i - \delta x^i dy_i \right. \\ \left. - \omega \delta \lambda du - \lambda \frac{\partial \omega}{\partial x^i} \delta x^i du - \lambda \frac{\partial \omega}{\partial y_i} \delta y_i du \right), \end{aligned} \quad (165)$$

and for an extremal we demand $\delta J = 0$ for *arbitrary* variations δx^i , δy_i , $\delta \lambda$, subject only to $\delta x^i = 0$ at the ends of I . Hence we obtain the equations of the extremals in the form

$$\frac{dx^i}{du} = \lambda \frac{\partial \omega}{\partial y_i}, \quad \frac{dy_i}{du} = -\lambda \frac{\partial \omega}{\partial x^i}, \quad \omega = 0. \quad (166)$$

The Lagrange function $\lambda(u)$ remains indeterminate; this corresponds to the indeterminacy of θ in (162). For any chosen function ω , we can choose the parameter u so that the equations of the extremal take the *Hamiltonian* or *canonical* form

$$\frac{dx^i}{du} = \frac{\partial \omega}{\partial y_i}, \quad \frac{dy_i}{du} = -\frac{\partial \omega}{\partial x^i}. \quad (167)$$

An extremal is determined by the values of (x^i, y_i) for $u = 0$, subject of course to $\omega(x, y) = 0$.

Let us now reinterpret the foregoing, using as geometrical background 4-dimensional space-time with coordinates x^i , instead of V_8 . We now regard y_i as a vector associated with the point x^i . The Hamiltonian surface Σ no longer appears as a 7-space immersed in V_8 , but as a set of 3-spaces, one such 3-space, say Y_3 , being associated with each point of space-time. We have to think, in fact, of a 4-dimensional y -space attached to each x -point, and we regard Y_3 as a 3-space in that y -space; its equation is $\omega(x, y) = 0$, with x^i held fixed. In the y -space the coordinates are y_i , and, in order that the theory may be invariant under arbitrary transformations of the x 's in space-time, it is necessary that $y_i dx^i$ shall be an invariant in (163); this means that y_i must transform like a *covariant* vector.

Viewed in V_8 , an extremal is a curve drawn on Σ . Viewed in space-time, it is a curve $x^i = x^i(u)$ bearing an associated vector field

$y_i = y_i(u)$. It should be clearly understood that we are dealing with one single mathematical theory; only the geometrical interpretation changes when we change our viewpoint from V_8 to space-time. Henceforth we shall use the space-time viewpoint, so that a *point* means a set of x -values.

Given two points, $A(x')$ and $B(x)$, there may happen to be no extremal joining them. (Actually, this is a rather exceptional case, but it should be borne in mind.) If such an extremal does exist, we write

$$S(x', x) = S(A, B) = \int_A^B y_i dx^i, \quad (168)$$

the integral being taken along the extremal. This is Hamilton's *principal* or *characteristic*¹ function. If we now vary A and B and compare the new extremal with the old one, we get from (165) (the integral vanishes for an extremal)

$$\delta S = y_i \delta x^i - y_{i'} \delta x^{i'}. \quad (169)$$

If the eight differentials δx^i , $\delta x^{i'}$ may be given arbitrary values (this is not always possible — we may not get varied points which can be joined by an extremal), then we have

$$\frac{\partial S}{\partial x^i} = y_i, \quad \frac{\partial S}{\partial x^{i'}} = -y_{i'}. \quad (170)$$

Substitution in $\omega(x, y) = 0$ then gives the *Hamilton-Jacobi equation*

$$\omega\left(x, \frac{\partial S}{\partial x}\right) = 0, \quad (171)$$

with of course a second equation corresponding to the second of (170).

Consider now a set of extremals in space-time, forming a domain D which may be of two, three or four dimensions. Throughout D there is a vector field y_i defined by the extremals, and we may speak meaningfully of the *circulation*

$$\kappa = \oint_C y_i dx^i \quad (172)$$

in any closed circuit C contained in D . We say that the set of extremals

¹ Not to be confused with the world-function Ω of Chap. II; in Riemannian space S and Ω are closely related, for we have $S^2 = |2\Omega|$.

form a *coherent system* if $\kappa = 0$ for all ¹ closed circuits C in D . The extremals forming a coherent system are called *rays*.

For a coherent system, let A be a fixed point in D and B a variable point in D . Then, since $\kappa = 0$, the integral

$$I(A, B) = \int_A^B y_i dx^i \quad (173)$$

is independent of the path of integration. If b is any constant, the equation

$$I(A, B) = b \quad (174)$$

limits B to a subspace of D ; this subspace is called a *wave*. By varying the constant b , we get a set of waves; it is easy to see that (since $\kappa = 0$) this set of waves is independent of the choice of A in D . In fact, the waves are the integrals of the total differential equation

$$y_i dx^i = 0. \quad (175)$$

It is to be particularly noted that every set of extremals does not define rays and waves; they are defined only for coherent systems, for which $\kappa = 0$ and hence (175) is integrable in D .

The simplest set of rays and waves is that associated with a set of extremals drawn from a fixed point A and forming a domain D . It is easy to show that this set is coherent. For if $S(A, B)$ is the principal function, then by (169) we have, for any closed circuit C in D ,

$$\oint_C y_i dx^i = \oint_C dS = 0. \quad (176)$$

The associated waves are

$$S(A, B) = \text{const.}, \quad (177)$$

where B is a variable point; or, to be more precise, the waves are the intersections of (177) with D .

The integral (173) for a coherent system is a function of B only since A is fixed. It is called the *one-point principal function* of the system; denoting it by $U(x)$, we may write the equations of the waves

$$U(x) = \text{const.} \quad (178)$$

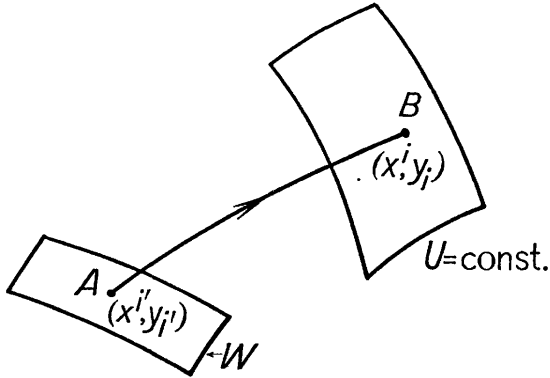
We shall now construct the most general coherent system, starting from some subspace W of space-time; W may be of zero, one, two, or

¹ For simplicity, we consider only the case where D is simply connected.

three dimensions. On W we assign some function of position U' , and then choose on W a vector field y_i , satisfying

$$\omega(x', y') = 0, \quad y_i \delta x^i = \delta U' \quad (179)$$

for every displacement δx^i in W . The choice of W and U' may be such that this is impossible, but we shall suppose it possible. The next step is to draw from the points of W extremals with the initial values (x^i, y_i) (Fig. 7). Then to any point $B(x)$ in the domain covered by these extremals we assign the function



$$U(x) = \int_A^B y_i dx^i + U'(A), \quad (180)$$

Fig. 7 — Construction of rays and waves from a subspace W

where A is the point at which the extremal through B leaves W , and

the integral is taken along the extremal. Varying B , we get

$$\delta U(x) = y_i \delta x^i - y_i \delta x^i + \delta U' = y_i \delta x^i \quad (181)$$

by (179). This is an exact differential in the domain covered by the extremals, so that the circulation κ vanishes. Therefore the extremals form a coherent system of rays, with waves given by $U(x) = \text{const.}$; $U(x)$ is in fact the one-point principal function, as in (178).

So much for general Hamiltonian theory. Let us now return to Riemannian space-time with tensor g_{ij} . As we have seen, Hamiltonian theory is based on the choice of a surface Σ with equation $\omega(x, y) = 0$. We can superimpose the two ideas; we have then two sets of important curves in space-time — geodesics and Hamiltonian extremals — with no obvious connection between the one and the other ¹.

On the other hand we can develop the theory of rays and waves in Riemannian space-time, using nothing but the tensor g_{ij} . This is done by taking for the Hamiltonian surface Σ the equation

$$\omega(x, y) = \omega_1(x, y)\omega_2(x, y)\omega_3(x, y) = 0, \quad (182)$$

where

$$\begin{aligned} \omega_1(x, y) &= g^{ij}y_i y_j, & \omega_2(x, y) &= g^{ij}y_i y_j + 1, \\ \omega_3(x, y) &= g^{ij}y_i y_j - 1. \end{aligned} \quad (183)$$

¹ In the geometrical optics of a medium (cf. Chap. XI-§ 2) a connection is set up. The Hamiltonian extremals are optical rays; the geodesics are of secondary importance.

Then the 3-space Y_3 for fixed x -values (so that g^{ij} are constants) is an algebraic surface of the sixth degree. It breaks up into sheets as shown in Fig. 8. We have the cone $\omega_1 = 0$, a cone with two sheets; the two-sheeted hyperboloid $\omega_2 = 0$; and the one-sheeted hyperboloid $\omega_3 = 0$. The y -vector must have its extremity on one of these sheets.

By (166) we have the following equations for the extremals:

$$\begin{aligned}\frac{dx^i}{du} &= 2\lambda g^{ij}y_j, \\ \frac{dy_i}{du} &= -\lambda g^{jk}_{,i}y_jy_k,\end{aligned}\tag{184}$$

where λ is some scalar; by changing the parameter u , we can change these equations into

$$\begin{aligned}\frac{dx^i}{du} &= g^{ij}y_j, \\ \frac{dy_i}{du} &= -\frac{1}{2}g^{jk}_{,i}y_jy_k.\end{aligned}\tag{185}$$

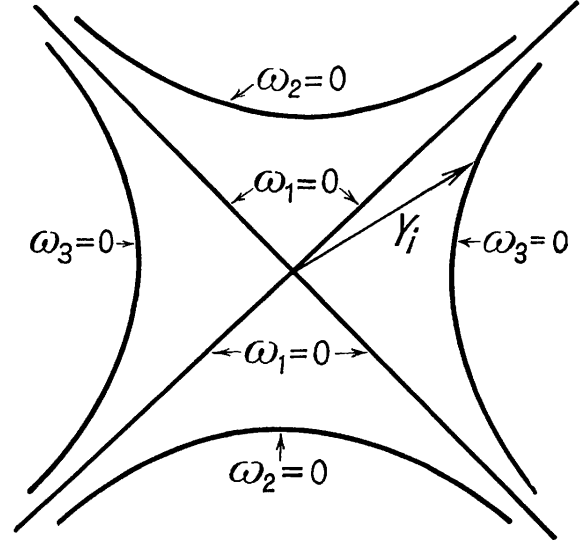


Fig 8 - The Hamiltonian surface Y_3 in y -space for rays and waves in Riemannian space-time

Eliminating the y 's we get by an easy calculation

$$\frac{\delta}{\delta u} \frac{dx^i}{du} = 0.\tag{186}$$

Further, the first of (185) gives

$$g_{ij} \frac{dx^i}{du} \frac{dx^j}{du} = g^{ij}y_iy_j \begin{cases} = 0 & \text{for } \omega_1 = 0, \\ = -1 & \text{for } \omega_2 = 0, \\ = 1 & \text{for } \omega_3 = 0. \end{cases}\tag{187}$$

Thus the parameter u in (185) and (186) is such that $du = ds$ for the last two cases.

We see at once from (186) that *the Hamiltonian extremals are geodesics*; those corresponding to $\omega_1 = 0$ are null geodesics, those corresponding to $\omega_2 = 0$ are timelike geodesics, and those corresponding to $\omega_3 = 0$ are spacelike geodesics.

As for Hamilton's principal function, we have by (185) and (187)

$$S(A, B) = \int_A^B y_i dx^i = \int_A^B g^{ij} y_i y_j du = \varepsilon \int_A^B ds \text{ or } 0, \quad (188)$$

according as the extremal is a timelike geodesic ($\varepsilon = -1$), a spacelike geodesic ($\varepsilon = 1$) or a null geodesic.

In a coherent system, *the rays are orthogonal to the waves*.¹ This is easy to show. By (175)

$$y_i \delta x^i = 0 \quad (189)$$

for every displacement δx^i in the wave. By (185)

$$y_i = g_{ij} \frac{dx^j}{du}. \quad (190)$$

Therefore

$$g_{ij} \delta x^i \frac{dx^j}{du} = 0, \quad (191)$$

which proves the statement.

Systems of null rays are of particular interest. For a null ray we have by (190)

$$y_i \frac{dx^i}{du} = g_{ij} \frac{dx^i}{du} \frac{dx^j}{du} = 0, \quad (192)$$

and so, by (189), null rays are not only orthogonal to waves — *they lie in them*.

A 3-space $f(x) = \text{const.}$ in space-time is called a *null surface* if f satisfies the partial differential equation

$$g^{ij} f_{|i} f_{|j} = 0, \quad (193)$$

which says essentially that the surface contains its own normal².

Let f satisfy (193). Then the equations

$$\frac{dx^i}{du} = g^{ij} f_{|j} \quad (194)$$

¹ In general Hamiltonian theory, (175) tells us that the *covariant* vector y_i is orthogonal to the wave, but until we introduce g_{ij} there can be no question of the orthogonality of the ray (dx^i/du , a *contravariant* vector) with the wave.

² The 3-space $x^4 = \text{const.}$ is a null surface if, and only if,

$$g^{44} = 0. \quad (193a)$$

define a set of null curves. Along such a curve we have

$$\frac{df}{du} = f_{|i} \frac{dx^i}{du} = f_{|i} g^{ij} f_{|j} = 0, \quad (195)$$

and therefore those curves (194) which start in $f(x) = 0$ remain in $f(x) = 0$. Further

$$\frac{\delta}{\delta u} \frac{dx^i}{du} = g^{ij} f_{|jk} \frac{dx^k}{du} = g^{ij} f_{|jk} g^{km} f_{|m}. \quad (196)$$

Now differentiation of (193) gives

$$g^{ij} f_{|i} f_{|jk} = 0, \quad (197)$$

or, since $f_{|jk} = f_{|kj}$,

$$g^{ij} f_{|i} f_{|kj} = 0. \quad (198)$$

Thus (196) gives

$$\frac{\delta}{\delta u} \frac{dx^i}{du} = 0, \quad (199)$$

and therefore the curves (194) are null geodesics. We may state then, as a theorem in the geometry of null surfaces, that *all the null geodesics drawn tangent to a null surface lie in that null surface, and, when their equations are written in the form (194), u is a special parameter*¹.

Let us apply this geometrical result to Hamiltonian theory, regarding the null geodesics (194) in $f(x) = 0$ as a system. Comparing (185) with (194), we have $y_i = f_{|i}$, and so

$$\oint_C y_i dx^i = \oint_C df = 0 \quad (200)$$

for every closed circuit in $f(x) = 0$. We have therefore a coherent system in which the rays are the null geodesics. Since

$$\int_A^B y_i dx^i = \int_A^B df = 0 \quad (201)$$

for every open curve in $f(x) = 0$, it follows that $f(x) = 0$ is a wave

¹ At each point on a null surface, the elementary null cone has one direction in common with the null surface and touches the null surface along that direction. The null geodesics (194) are formed out of the elementary vectors of tangency.

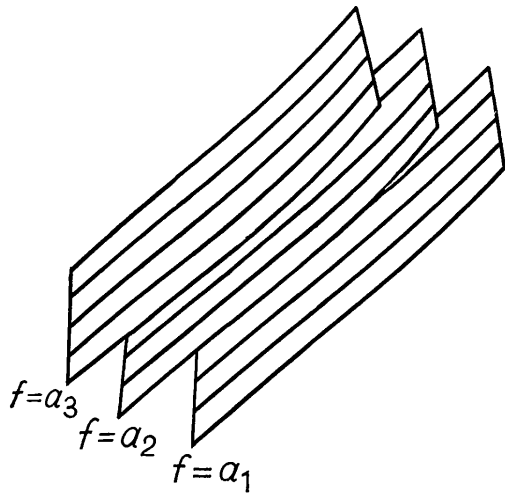


Fig. 9 — Null waves and the null rays contained in them

(in fact the only wave) associated with this system of null rays; we call it a *null wave*. It is clear that the equation $f(x) = \text{const.}$, with $f(x)$ satisfying (193), defines a set of null waves, as illustrated in Fig. 9 — this picture is only suggestive, for it should show ∞^2 null rays in each null wave.

Let us now briefly consider, as at (180), the construction of a coherent system, starting from a subspace W on which a function U' is assigned.

Consider first the case where W is a single point in space-time. Then U' is a mere constant, and the second condition in (179) disappears. We have only to satisfy $\omega(x', y') = 0$, and this we do by taking the vector y_i , with its extremity on one of the surfaces $\omega_1 = 0, \omega_2 = 0, \omega_3 = 0$ of Fig. 8; the corresponding extremals are null geodesics, timelike geodesics and spacelike geodesics, respectively. For the null geodesics we have $y_i dx^i = 0$ along the rays, and so (180) gives simply $U(x) = U'$, a constant. Since the waves are given by $U(x) = \text{const.}$, we see that the totality of null geodesics from the fixed point W form a single wave, the null cone (a particular case of a null surface, as considered above). In the cases of timelike and spacelike rays, we get the waves by taking constant measure along the rays, thus obtaining pseudospheres (rather like hyperboloids in ordinary space). The whole pattern of rays and waves for the case where W is a single point is illustrated in Fig. 10.

Let us close this discussion of rays and waves in space-time by taking the case where W is a timelike curve with equation $x^{i'} = x^{i'}(v)$. We are to select some function $U'(v)$, and then choose $y_{i'}$ to satisfy

$$\omega(x', y') = 0, \quad y_{i'} \frac{dx^{i'}}{dv} = \frac{dU'}{dv}. \quad (202)$$

Viewed in the y -space of Fig. 8, the second equation represents a 3-flat, and this 3-flat will cut one or more sheets of the ω -surfaces, yielding the required y -vectors. Let us, for definiteness, consider only intersections of the 3-flat with $\omega_1 = 0$, which intersections correspond to null rays. In general we shall get such intersections, and hence a system of null rays and waves as illustrated in Fig. 11, in which the null

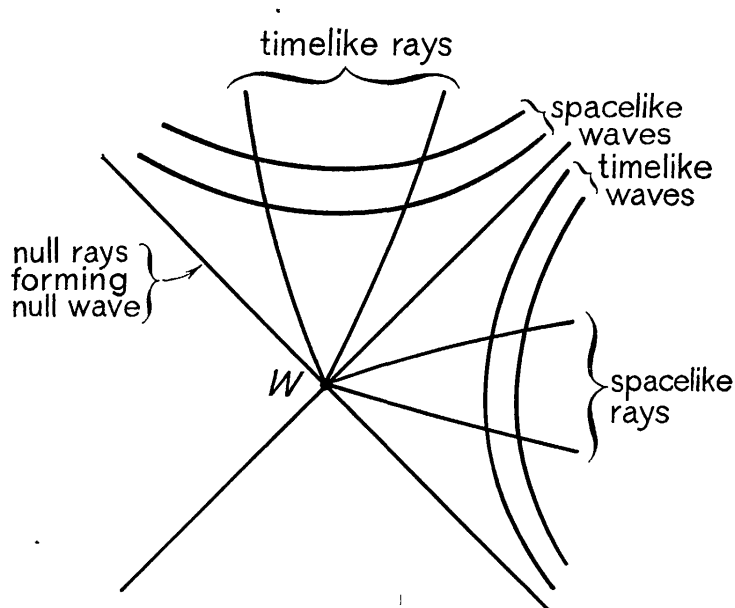


Fig. 10 – Rays and waves from a single point W in space-time

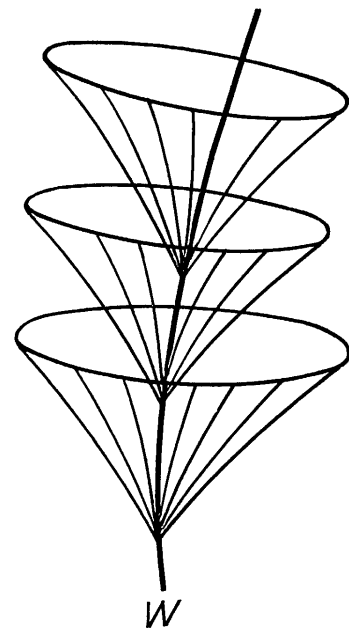


Fig. 11 – Rays and waves from a timelike curve W in space-time

waves are shown as if they were 2-dimensional (they are actually 3-dimensional). There is one exceptional case, namely, the case where we choose U' to be a constant. Then the second of (202) cannot be satisfied by any γ -vector with its extremity on $\omega_1 = 0$ (this is essentially due to the fact that no null vector can be orthogonal to a timelike vector).

§ 8. GAUSSIAN COORDINATES

Let x^i be admissible coordinates (§ 1) in space-time, and let Σ be a smooth 3-space defined by equations $x^i = f^i(\xi)$, where ξ stands for the three parameters ξ^ρ (Greek suffixes run 1, 2, 3). Let $U^i(\xi)$ be a vector field defined smoothly over Σ . Through the points of Σ we draw, in both directions, geodesics tangent to U^i ; let u be that unique special parameter on each geodesic such that $u = 0$, $dx^i/du = U^i$ on Σ . Let B be any point in the neighbourhood of Σ . Let A be the point where the geodesic through B meets Σ (Fig. 12). Then to B we attach the four *Gaussian coordinates* (u, ξ) where u is evaluated at B and ξ^ρ are evaluated at A . Where necessary, we distinguish between *normal* Gaussian coordinates (U^i orthogonal

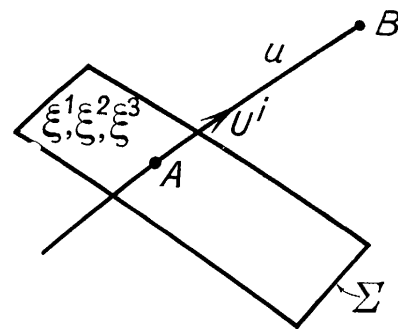


Fig. 12 – Gaussian coordinates \bar{x}^i . ($\bar{x}^\rho = \xi^\rho$, $\bar{x}^4 = u$)

to Σ) and *skew* Gaussian coordinates (U^i not orthogonal to Σ). We may also write \bar{x}^i for the Gaussian coordinates, putting

$$\bar{x}^\rho = \xi^\rho, \quad \bar{x}^4 = u. \quad (203)$$

Gaussian coordinates have two important features. First, they are (as we shall show) admissible coordinates, and so, since they are given by a simple geometrical construction, they lend concreteness to what might otherwise seem a purely formal definition of admissibility. Secondly, their use sometimes simplifies calculations.

To prove the admissibility of Gaussian coordinates, we note that on any one of the geodesics we have, in the original postulated admissible coordinates,

$$\frac{\delta}{\delta u} \frac{dx^i}{du} = \frac{d^2x^i}{du^2} + \Gamma_{jk}^i \frac{dx^j}{du} \frac{dx^k}{du} = 0, \quad (204)$$

and so

$$x^i = (f^i)_A + u(U^i)_A - \frac{1}{2}u^2(\Gamma_{jk}^i U^j U^k)_A + \dots, \quad (205)$$

the subscript A indicating evaluation at that point. Since x^i are admissible, the Γ 's are continuous across Σ , and so the terms written explicitly in (205) do not depend on the side of Σ which we approach in the limit $u \rightarrow 0$. That may not be true of the higher terms in the series, because admissibility of x^i does not imply continuity of the derivatives of the Γ 's, since these involve the second derivatives of the g_{ij} . But the terms shown explicitly in (205) are all we need. From them we calculate the following values for $u = 0$:

$$\begin{aligned} \frac{\partial x^i}{\partial \bar{x}^\rho} &= \frac{\partial f^i}{\partial \bar{x}^\rho}, & \frac{\partial x^i}{\partial \bar{x}^4} &= U^i, \\ \frac{\partial^2 x^i}{\partial \bar{x}^\rho \partial \bar{x}^\sigma} &= \frac{\partial^2 f^i}{\partial \bar{x}^\rho \partial \bar{x}^\sigma}, & \frac{\partial^2 x^i}{\partial \bar{x}^\rho \partial \bar{x}^4} &= \frac{\partial U^i}{\partial \bar{x}^\rho}, & \frac{\partial^2 x^i}{(\partial \bar{x}^4)^2} &= -\Gamma_{jk}^i U^j U^k. \end{aligned} \quad (206)$$

All these quantities are continuous across Σ ; in other words, the transformation $x \rightarrow \bar{x}$ is C^2 . Hence, from the formulae of transformation

$$\begin{aligned} \bar{g}_{ij} &= g_{ab} \frac{\partial x^a}{\partial \bar{x}^i} \frac{\partial x^b}{\partial \bar{x}^j}, \\ \bar{\Gamma}_{jk}^i &= \Gamma_{bc}^a \frac{\partial \bar{x}^i}{\partial x^a} \frac{\partial x^b}{\partial \bar{x}^j} \frac{\partial x^c}{\partial \bar{x}^k} - \frac{\partial^2 \bar{x}^i}{\partial x^a \partial x^b} \frac{\partial x^a}{\partial \bar{x}^j} \frac{\partial x^b}{\partial \bar{x}^k}, \end{aligned} \quad (207)$$

it follows that \bar{g}_{ij} and $\bar{\Gamma}_{jk}^i$ are continuous across Σ . Therefore, by (8), \bar{g}_{ij} are C^1 , which means that \bar{x}^i are admissible coordinates, and so the result is proved: Gaussian coordinates are admissible.

Passing now to the second feature of Gaussian coordinates, let us write the differential equations (204) of a geodesic in terms of the Gaussian coordinates \bar{x}^i . Since $\bar{x}^\rho = \text{const.}$, $\bar{x}^4 = u$, the equations reduce to

$$\bar{\Gamma}_{44}^i = 0, \quad (208)$$

or equivalently

$$2 \frac{\partial \bar{g}_{i4}}{\partial \bar{x}^4} = \frac{\partial \bar{g}_{44}}{\partial \bar{x}^i}. \quad (209)$$

Two cases arise. First, suppose that the vectors U^i are null. Then the geodesics used in the Gaussian construction are null, and we have $\bar{g}_{44} = 0$, so that, by (209),

$$\frac{\partial \bar{g}_{\rho 4}}{\partial \bar{x}^4} = 0, \quad \bar{g}_{44} = 0. \quad (210)$$

Secondly, suppose that the vectors U^i are all timelike or all spacelike. Let us normalise them by

$$g_{ij}U^iU^j = \varepsilon = \pm 1. \quad (211)$$

Then we have $du = ds$ on the geodesics, and $\bar{g}_{44} = \varepsilon$, so that, by (209),

$$\frac{\partial \bar{g}_{\rho 4}}{\partial \bar{x}^4} = 0, \quad \bar{g}_{44} = \varepsilon. \quad (212)$$

In the particular case of *normal* Gaussian coordinates, we have

$$\bar{g}_{\rho 4} = 0, \quad \bar{g}_{44} = \varepsilon. \quad (213)$$

It is possible to base Gaussian coordinates on a point, a curve, or a 2-space, instead of on a 3-space Σ as above. But we shall not fill in the details of this.

The formulae for the Riemann tensor, the Ricci tensor and the Einstein tensor are somewhat simplified by the use of normal Gaussian coordinates. Dropping the bars on the Gaussian coordinates, and not bothering about the geometrical construction for them, we may state the results briefly as follows.

Let there be a system of coordinates x^i for which the metric

form reads

$$\Phi = g_{\mu\nu}dx^\mu dx^\nu + \varepsilon(dx^4)^2, \quad \varepsilon = \pm 1. \quad (214)$$

Then we have

$$\begin{aligned} g_{\rho 4} &= 0, & g_{44} &= \varepsilon, \\ g^{\rho 4} &= 0, & g^{44} &= \varepsilon, \\ g^{\rho\alpha}g_{\rho\beta} &= \delta^\alpha_\beta. \end{aligned} \quad (215)$$

Then from the formulae of § 5 we obtain the following expressions:

$$\begin{aligned} R_{\rho\mu\nu\sigma} &= \bar{R}_{\rho\mu\nu\sigma} + \frac{1}{4}\varepsilon(g_{\rho\sigma,4}g_{\mu\nu,4} - g_{\rho\nu,4}g_{\sigma\mu,4}), \\ R_{\rho\mu 4\sigma} &= \frac{1}{2}(D_\mu g_{\rho\sigma,4} - D_\rho g_{\mu\sigma,4}), \\ R_{\rho 44\sigma} &= R_{4\rho\sigma 4} = \frac{1}{2}g_{\rho\sigma,44} - \frac{1}{4}g^{\alpha\beta}g_{\rho\alpha,4}g_{\sigma\beta,4}; \end{aligned} \quad (216)$$

$$\begin{aligned} R_{\mu\nu} &= \bar{R}_{\mu\nu} + \frac{1}{2}\varepsilon g_{\mu\nu,44} + \frac{1}{4}\varepsilon A g_{\mu\nu,4} - \frac{1}{2}\varepsilon g^{\alpha\beta}g_{\mu\alpha,4}g_{\nu\beta,4}, \\ R_{\mu 4} &= \frac{1}{2}g^{\rho\sigma}(D_\mu g_{\rho\sigma,4} - D_\rho g_{\mu\sigma,4}) = \frac{1}{2}A_{,\mu} - \frac{1}{2}D^\sigma g_{\mu\sigma,4}, \\ R_{44} &= \frac{1}{2}C - \frac{1}{4}B; \end{aligned} \quad (217)$$

$$R = \bar{R} + \varepsilon(\frac{1}{4}A^2 - \frac{3}{4}B + C); \quad (218)$$

$$\begin{aligned} G_{\mu\nu} &= \bar{G}_{\mu\nu} + \frac{1}{2}\varepsilon g_{\mu\nu,44} + \frac{1}{4}\varepsilon A g_{\mu\nu,4} - \frac{1}{2}\varepsilon g^{\alpha\beta}g_{\mu\alpha,4}g_{\nu\beta,4} \\ &\quad - \varepsilon g_{\mu\nu}(\frac{1}{8}A^2 - \frac{3}{8}B + \frac{1}{2}C), \\ G_{\mu 4} &= R_{\mu 4} = \frac{1}{2}A_{,\mu} - \frac{1}{2}D^\sigma g_{\mu\sigma,4}, \\ G_{44} &= -\frac{1}{2}\varepsilon\bar{R} - \frac{1}{8}A^2 + \frac{1}{8}B. \end{aligned} \quad (219)$$

The explanation of the symbols is as follows:

$$\begin{aligned} \bar{R}_{\rho\mu\nu\sigma} &= \text{Riemann subtensor}^1 \text{ of } x^4 = \text{const.}, \\ \bar{R}_{\mu\nu} &= \text{Ricci subtensor of } x^4 = \text{const.}, \\ \bar{R} &= \text{curvature subinvariant of } x^4 = \text{const.}, \\ \bar{G}_{\mu\nu} &= \text{Einstein subtensor of } x^4 = \text{const.}, \\ D_\mu &= \text{operator of covariant differentiation in } x^4 = \text{const.} \\ &\quad \text{with respect to subtensor } g_{\alpha\beta}; \end{aligned} \quad (220)$$

$$A = g^{\mu\nu}g_{\mu\nu,4}, \quad B = g^{\mu\nu}g^{\rho\sigma}g_{\mu\rho,4}g_{\nu\sigma,4}, \quad C = g^{\mu\nu}g_{\mu\nu,44}. \quad (221)$$

¹ Each 3-space $x^4 = \text{const.}$ is Riemannian with fundamental tensor $\bar{g}_{\alpha\beta} = g_{\alpha\beta}$ and conjugate $\bar{g}^{\alpha\beta} = g^{\alpha\beta}$; cf. (215). The barred quantities are calculated in each such 3-space in the usual way, using these tensors; they are called *subtensors* because they are not tensors for general transformations of x^i , but only for transformations of x^δ , with x^4 untransformed. Cf. SYNGE and SCHILD [1956], p. 67.

§ 9. JUNCTION CONDITIONS ACROSS A 3-SPACE OF DISCONTINUITY

We recall from § 1 the assumed existence of admissible coordinates in space-time, for which coordinates we have continuity of g_{ij} and $g_{ij,k}$ across any 3-space Σ . If Σ is in some sense a 3-space of discontinuity, the discontinuity can occur only in the second or higher derivatives of g_{ij} , provided the coordinates are admissible. We proceed to examine the situation.

Given Σ (it might be a null surface), let us transform from the original admissible coordinates to new admissible coordinates¹ such that the equation of Σ becomes $x^4 = 0$. Then the following quantities are continuous across Σ :

$$g_{ij}, \quad g^{ij}, \quad g_{ij,k}, \quad [ij, k], \quad \Gamma_{jk}^i, \quad g_{ij,k}, \quad (222)$$

since these involve at most one differentiation with respect to x^4 (as usual, Latin suffixes run 1, 2, 3, 4 and Greek 1, 2, 3). Also $g_{ij,k\alpha}$ are continuous, but we may expect discontinuities in $g_{ij,44}$.

Let us use the symbol $[C]$ to indicate any quantity which is continuous across Σ . Then by (85) and (222) we have

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= [C], & R_{\alpha\beta 4\delta} &= [C], \\ R_{\alpha 44\beta} &= R_{4\alpha\beta 4} = \frac{1}{2}g_{\alpha\beta,44} + [C]. \end{aligned} \quad (223)$$

Hence

$$\begin{aligned} R_{\alpha\beta} &= g^{ij}R_{i\alpha\beta j} = g^{44}R_{4\alpha\beta 4} + [C] = \frac{1}{2}g^{44}g_{\alpha\beta,44} + [C], \\ R_{\alpha 4} &= g^{ij}R_{i\alpha 4j} = g^{4\beta}R_{4\alpha 4\beta} + [C] = -\frac{1}{2}g^{4\beta}g_{\alpha\beta,44} + [C], \\ R_{44} &= g^{ij}R_{i44j} = g^{\alpha\beta}R_{\alpha 44\beta} = \frac{1}{2}g^{\alpha\beta}g_{\alpha\beta,44} + [C], \end{aligned} \quad (224)$$

and

$$\begin{aligned} R &= g^{\alpha\beta}R_{\alpha\beta} + 2g^{\alpha 4}R_{\alpha 4} + g^{44}R_{44} \\ &= (g^{\alpha\beta}g^{44} - g^{\alpha 4}g^{4\beta})g_{\alpha\beta,44} + [C]. \end{aligned} \quad (225)$$

Then

$$\begin{aligned} R_{\alpha}^4 &= g^{4i}R_{\alpha i} = g^{4\beta}R_{\alpha\beta} + g^{44}R_{\alpha 4} = [C], \\ R_4^4 &= g^{4i}R_{4i} = g^{4\alpha}R_{4\alpha} + g^{44}R_{44} \\ &= \frac{1}{2}(g^{44}g^{\alpha\beta} - g^{4\alpha}g^{4\beta})g_{\alpha\beta,44} + [C], \end{aligned} \quad (226)$$

and so

$$G_{\alpha}^4 = R_{\alpha}^4 = [C], \quad G_4^4 = R_4^4 - \frac{1}{2}R = [C], \quad (227)$$

or

$$G_i^4 = [C]. \quad (228)$$

¹ Possibly, but not necessarily, Gaussian coordinates (skew-Gaussian if Σ is null).

This is a set of four *junction conditions*: the mixed components G_i^4 of the Einstein tensor are continuous across $x^4 = 0$, provided the coordinates are admissible. This is equivalent to saying that, as a mere matter of direct calculation, the components G_i^4 do not contain second derivatives with respect to x^4 .

We now wish to pass to other systems of admissible coordinates, and it is convenient to write \bar{x}^i for the admissible coordinates used above, for which the equation of Σ is $\bar{x}^4 = 0$. For any other admissible coordinates x^i , let the equation of Σ be $f(x) = 0$. Then $f_{,i}$ is a covariant normal to Σ , and $G_i^j f_{,j}$ is a vector which in the \bar{x} -coordinates has the value \bar{G}_i^4 , and is, as we have seen, continuous across Σ . Since the transformation $x \rightarrow \bar{x}$ is C^2 , the junction conditions now read as follows: *for any admissible coordinates*,

$$G_i^j f_{,j} = [C], \quad (229)$$

i.e. is continuous across Σ .

We have released ourselves from those special admissible coordinates for which the equation of Σ is $\bar{x}^4 = 0$. We now go a step further in emancipating the coordinate system. Let \bar{x}^i be any admissible coordinates and let x^i be new coordinates (no longer admissible) obtained from \bar{x}^i by a transformation that is only C^1 . The components g_{ij} are still continuous across Σ (since their transformation law involves only the first derivatives $\partial x^i / \partial \bar{x}^j$), but the first derivatives $g_{ij,k}$ may now be discontinuous. However $G_i^j f_{,j}$ is a vector and its transformation involves only the first derivatives $\partial x^i / \partial \bar{x}^j$. Therefore we have this result ¹: *For coordinates obtained from admissible coordinates by C^1 transformation, the junction condition (229) holds.*

Finally, by working with an invariant, we effect a complete divorce of the coordinate systems on the two sides of Σ . Let \bar{x}^i be any admissible coordinates and $\bar{\phi}^i$ a contravariant vector field which undergoes parallel transport along any assigned set of curves which cross Σ . We pass to new coordinates x^i by piecewise smooth transformations on the two sides of Σ , but without demanding even continuity of the transformation across Σ (the x 's may be discontinuous functions of the \bar{x} 's across Σ). Now

$$I = G_i^j f_{,j} \bar{\phi}^i \quad (230)$$

¹ Cf. ISRAEL [1958]. Much of the work done on junction conditions prior to the introduction of admissible coordinates by LICHNEROWICZ [1955a] is mathematically obscure.

is an invariant, and it is certainly continuous across Σ when we use the admissible coordinates. Therefore it is continuous when we use the coordinates x^i . The junction condition now reads: *I is continuous across Σ for the independent coordinate systems just described, ϕ^i undergoing parallel transport across Σ .* On account of the arbitrariness in the choice of ϕ^i , we have actually four conditions here, as in (228) or (229). The equation of Σ is $f(x) = 0$, but now the form of the function f may be quite different on the two sides of Σ .

If we prefer, we can use a covariant vector ϕ_i , so that the continuous invariant reads

$$I = G^{ij} f_{,i} \phi_j. \quad (231)$$

§ 10. THEOREMS OF STOKES AND GREEN

In Euclidean 3-space the theorem of Stokes expresses an integral taken round a closed curve C as an integral taken over a surface S which spans C , and is usually written

$$\oint_C (u dx + v dy + w dz) = \iint \left[\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) l + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) m + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) n \right] dS, \quad (232)$$

(u, v, w) being a vector field and (l, m, n) the direction cosines of the normal to S , so directed that the sense of integration on C bears to this direction the same relation that a rotation of the x -axis into the y -axis bears to the z -axis. Green's theorem (also called Gauss' theorem) expresses an integral taken over a closed surface S as an integral taken over the volume bounded by S , and is usually written

$$\iint (lu + mv + nw) dS = \iiint \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) dV, \quad (233)$$

where (l, m, n) are the direction cosines of the outward normal to S .

The above formulae can be generalized. It is remarkable that actually Stokes' theorem does not essentially involve a metric tensor g_{ij} , and that, when a metric is introduced, Green's theorem appears as a particular case of Stokes' theorem. It makes for clarity to present first the generalized formulae in an N -space V_N without a metric, then take $N = 4$ for space-time (still without a metric), and finally

introduce g_{ij} . The formulae will be stated and explained, but their proofs will not be given here ¹.

Consider an N -space V_N with coordinates x^i , but without (for the present) a metric g_{ij} . In V_N take a subspace V_M where $M = 1, 2, \dots$ up to $N - 1$, or even N , in which last case V_M is V_N itself. Using Latin suffixes for the range $1, 2, \dots, N$ and Greek suffixes for $1, 2, \dots, M$, we write $x^i = x^i(y)$ as the equations of V_M , where y stands for M parameters y^ρ .

Consider a *cell* in V_M with M ordered edges $d_\rho y^\sigma$, where $\rho (=1, 2, \dots, M)$ are labels enumerating the several edges. Write

$$\det(d_\rho y^\sigma) = \Delta. \quad (234)$$

The M -cell is *positively* or *negatively oriented* relative to the coordinate system y according as Δ is positive or negative. The *tensor extension* of the cell is defined as

$$d\tau^{i_1 i_2 \dots i_M} = \varepsilon_{\alpha_1 \alpha_2 \dots \alpha_M} \frac{\partial x^{i_1}}{\partial y^{\alpha_1}} \frac{\partial x^{i_2}}{\partial y^{\alpha_2}} \dots \frac{\partial x^{i_M}}{\partial y^{\alpha_M}} \Delta, \quad (235)$$

where the ε -symbol is the M -dimensional numerical permutation symbol ². Obviously this expression (235) is a contravariant tensor with respect to x -transformations, and it is skew symmetric in each pair of indices. It is an invariant with respect to y -transformations.

In two special cases, (235) is very simple. First take $M = 1$; the cell is 1-dimensional, and the extension degenerates into

$$d\tau^i = dx^i. \quad (236)$$

Now take $M = N$ and $y^i = x^i$; we get

$$d\tau^{i_1 i_2 \dots i_N} = \varepsilon_{i_1 i_2 \dots i_N} \Delta, \quad \Delta = \det(dx^i), \quad (237)$$

and if we take the cell with edges along the parametric lines of the coordinates, we get

$$\Delta = dx^1 dx^2 \dots dx^N. \quad (238)$$

We now consider an open V_M immersed in V_N ; let V_{M-1} be the closed $(M - 1)$ -space which bounds V_M . Let $T_{i_1 \dots i_{M-1}}$ be a covariant tensor field defined over V_M and its neighbourhood. Then the *generalized*

¹ Cf. PAULI [1958], p. 52, SCHOUTEN [1954], p. 97; and, for a more detailed treatment, SYNGE and SCHILD [1956], p. 274.

² Cf. § 5 for $M = 4$; the generalization is obvious.

Stokes' theorem reads (we merely state it here without proof)

$$\oint_{V_{M-1}} T_{i_1 \dots i_{M-1}} d\tau^{i_1 \dots i_{M-1}} = \int_{V_M} T_{i_1 \dots i_{M-1}, i_M} d\tau^{i_1 \dots i_M}, \quad (239)$$

the comma indicating a partial derivative. The left hand side is obviously an invariant. The right hand side is also an invariant, as is easily verified when the skew symmetry of the element of extension is taken into account.

It must be noted however that (239) is true only when the orientations of the cells are properly chosen — otherwise a minus sign will appear. To enlarge on this point, it must be understood that V_{M-1} and V_M are *orientable spaces*. This means (to explain it for V_M) that if we carry an M -cell round any closed circuit in V_M , without letting Δ vanish, then, on completion of the circuit, the new set of ordered edges can be changed continuously into the old set, without letting Δ vanish. ($\Delta = 0$ means that the M -cell degenerates into an $(M - 1)$ -cell.) To illustrate by surfaces ($M = 2$) in ordinary Euclidean space, the surface of a sphere is orientable, and so is the surface of a torus, whereas a Möbius band is not. Given, then, that V_{M-1} and V_M are orientable spaces, in order that (239) may hold (and not the same equation with a minus sign inserted before one of the integrals), it is necessary that we should be able to reconcile the set of ordered edges of an M -cell of V_M with the set of ordered edges of an $(M - 1)$ -cell of V_{M-1} plus (as a *last edge*) an element pointing *out* from V_M across V_{M-1} . This is illustrated in Fig. 13. This rule of orientation

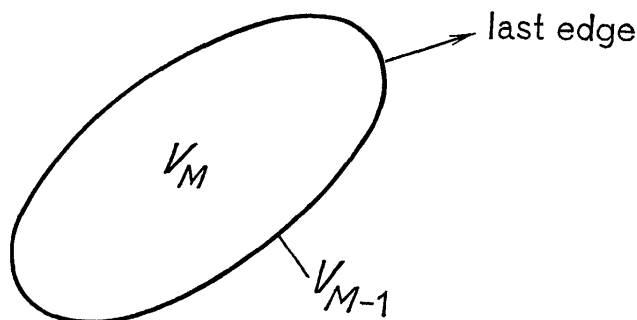


Fig. 13 – Direction of the last edge to be added to $(M - 1)$ -cell to make M -cell

is involved in the elementary formula (232), but obscured somewhat by the fact that one integral is taken along a curve.

We return to the general Stokes' formula (239) and put $N = 4$, so that $V_N = V_4 = \text{space-time}$, but still without a metric tensor. Setting in turn $M = 2, 3$ and 4 , we obtain the three following forms of

Stokes' theorem:

$$\oint_{V_1} T_i dx^i = \int_{V_2} T_{i,j} d\tau^{ij}, \quad (240)$$

$$\oint_{V_2} T_{ij} d\tau^{ij} = \int_{V_3} T_{ij,k} d\tau^{ijk}, \quad (241)$$

$$\oint_{V_3} T_{ijk} d\tau^{ijk} = \int_{V_4} T_{ijk,m} d\tau^{ijkm}, \quad (242)$$

where, in the last integral, V_4 represents the portion of space-time enclosed by V_3 .

It is not assumed in the above formulae that T_{ij} and T_{ijk} have any special symmetries; however, on account of the skew-symmetry of the tensor extensions, only the skew-symmetric parts of the T 's actually remain in the formulae.

Finally, we introduce the tensor g_{ij} of space-time. It is now permissible to replace the partial derivatives by covariant derivatives, because the additional terms drop out. For example,

$$T_{ij|k} d\tau^{ijk} = (T_{ij,k} - \Gamma_{ik}^a T_{aj} - \Gamma_{jk}^a T_{ia}) d\tau^{ijk} = T_{ij,k} d\tau^{ijk}. \quad (243)$$

Accordingly we can display the Stokes' formulae in completely invariant form as follows:

$$\oint_{V_1} T_i dx^i = \int_{V_2} T_{i|j} d\tau^{ij}, \quad (244)$$

$$\oint_{V_2} T_{ij} d\tau^{ij} = \int_{V_3} T_{ij|k} d\tau^{ijk}, \quad (245)$$

$$\oint_{V_3} T_{ijk} d\tau^{ijk} = \int_{V_4} T_{ijk|m} d\tau^{ijkm}. \quad (246)$$

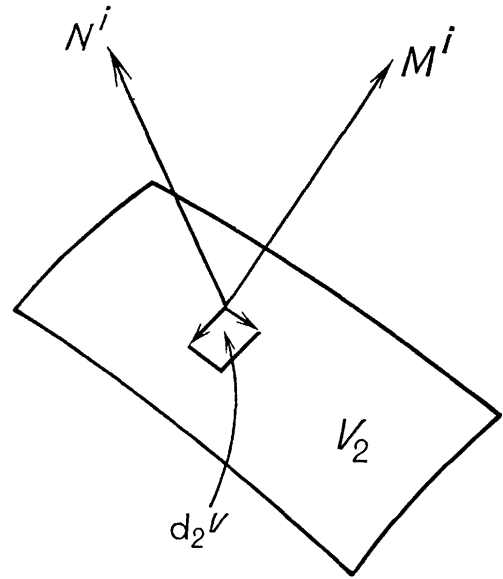
We shall now display these formulae in a different way, introducing the invariant element of volume (2-, 3- or 4-dimensional), defined as the product of the measures (ds) of the edges of a rectangular cell.

Consider V_2 . Let M^i and N^i be unit vectors, orthogonal to V_2 and to one another (Fig. 14). Then the tensor extension of a 2-cell may be written

$$d\tau^{ij} = \varepsilon(M)\varepsilon(N)\eta^{ijkm}M_kN_md_2v, \quad (247)$$

where the ε 's are the indicators of the vectors, the η -symbol is the permutation tensor as in (114), and d_2v is the invariant 2-volume of the cell; the 2-cell is such that its ordered edges, together with M^i and N^i in order, form a tetrad with the same orientation as the parametric

lines of the coordinates x^i . We have to prove (247), and this we do by recognizing that it is a tensor formula, and so may be verified by using a special coordinate system. It suffices to consider a rectangular cell, since an oblique cell may be broken up into a great number of small rectangular cells. Let us then use coordinates for which at the point in question g_{ij} is the diagonal matrix with elements $(1, 1, 1, -1)$ in some order, and for which the parametric lines of x^1 and x^2 run along the edges of the cell, while those of x^3 and x^4 lie along M^i and N^i respectively. We take $y^1 = x^1$, $y^2 = x^2$. Then by (234) we have $\Delta = dx^1 dx^2$, and by (235)

Fig. 14 – 2-cell on V_2

$$d\tau^{12} = -d\tau^{21} = dx^1 dx^2, \quad (248)$$

while the other components of $d\tau^{ij}$ vanish. On the other hand, we have $M_3 = \varepsilon(M)$, $N_4 = \varepsilon(N)$, and the other covariant components vanish. Thus the right hand side of (247) survives only if $i = 1, j = 2$ or $i = 2, j = 1$, and for the former values it is equal to d_2v . But $d_2v = dx^1 dx^2$, and so, on comparing this with (248), we verify (247).

Likewise for V_3 we have

$$d\tau^{ijk} = \varepsilon(N)\eta^{ijkm}N_md_3v, \quad (249)$$

where N^i is a unit vector normal to V_3 , such that when it is associated as a fourth direction with the ordered edges of the 3-cell, we get a 4-cell with the same orientation as the parametric lines of the coordinates; d_3v is the invariant 3-volume of the cell. For V_4 we have

$$d\tau^{ijkm} = \eta^{ijkm}d_4v, \quad (250)$$

where d_4v is the invariant 4-volume, provided the 4-cell has the same orientation as the parametric lines of the coordinates — otherwise a minus sign must be inserted.

Since the covariant derivative of the permutation tensor vanishes,

we may now write the Stokes formulae (244)–(246) as follows:

$$\oint_{V_1} T_i dx^i = \int_{V_2} (T_i \eta^{ijkm})_{|j} \varepsilon(M) \varepsilon(N) M_k N_m d_2 v, \quad (251)$$

$$\oint_{V_2} T_{ij} d\tau^{ij} = \int_{V_3} (T_{ij} \eta^{ijkm})_{|k} \varepsilon(N) N_m d_3 v, \quad (252)$$

$$\oint_{V_3} T_{ijk} d\tau^{ijk} = \int_{V_4} (T_{ijk} \eta^{ijkm})_{|m} d_4 v. \quad (253)$$

The left hand sides of (252) and (253) might also be written in forms involving invariant elements of volume. If we use (249) in (253), we get

$$\oint_{V_3} T_{ijk} \eta^{ijkm} \varepsilon(N) N_m d_3 v = \int_{V_4} (T_{ijk} \eta^{ijkm})_{|m} d_4 v. \quad (254)$$

Now, given any vector U^i , we define its *dual* by

$$U_{ijk}^* = \eta_{ijkm} U^m, \quad (255)$$

and this gives

$$U^m = \frac{1}{6} \eta^{ijkm} U_{ijk}^*. \quad (256)$$

Therefore, for any vector field U^i , (254) gives

$$\oint_{V_3} U^i \varepsilon(N) N_i d_3 v = \int_{V_4} U^i_{|i} d_4 v. \quad (257)$$

This is the *generalization of the theorem of Green (or Gauss)*. Note that N^i points out of the domain V_4 across V_3 ; note also the presence of the indicator $\varepsilon(N)$.

If we contract V_1 to a point in (240) or (244) or (251), the integral on the left hand side vanishes, and so we get a vanishing integral over a closed V_2 . Likewise, we may contract V_2 to a point in (241) or (245) or (252). But unless space-time were multiply connected, it would be futile to contract V_3 to a point in (242) or (246) or (253), because this would make the domain V_4 contract to a point at the same time. By the above process of contraction to a point, we obtain the following identities for closed subspaces in space-time:

$$\text{without metric: } \left\{ \begin{array}{l} \oint_{V_2} T_{i,j} d\tau^{ij} = 0, \\ \oint_{V_3} T_{ij,k} d\tau^{ijk} = 0; \end{array} \right\} \quad (258)$$

$$\text{with metric: } \left\{ \begin{array}{l} \oint_{V_2} T_{i|j} d\tau^{ij} = 0, \\ \oint_{V_2} (T_i \eta^{ijkm})_{|j} \varepsilon(M) \varepsilon(N) M_k N_m d_2 v = 0, \\ \oint_{V_3} T_{ij|k} d\tau^{ijk} = 0, \\ \oint_{V_3} (T_{ij} \eta^{ijkm})_{|k} \varepsilon(N) N_m d_3 v = 0. \end{array} \right\} \quad (259)$$

CHAPTER II

THE WORLD-FUNCTION Ω

§ 1. THE WORLD-FUNCTION Ω AND ITS COVARIANT DERIVATIVES AS A TWO-POINT INVARIANT AND TWO-POINT TENSORS

Let $P'(x')$ and $P(x)$ be two points of space-time, joined by a geodesic Γ with equations $x^i = \xi^i(u)$ where u is a special parameter [cf. I-§ 2]. Then the integral [cf. I-(27)]

$$\Omega(P'P) = \Omega(x', x) = \frac{1}{2}(u_1 - u_0) \int_{u_0}^{u_1} g_{ij} U^i U^j du, \quad (1)$$

taken along Γ with $U^i = d\xi^i/du$, has a value independent of the particular special parameter chosen. If, as we shall suppose, the points P', P determine a unique geodesic passing through them, then Ω is a function of these two points; it is a function of the eight variables $x^{i'}$, x^i , and we shall call it the *world-function*¹ of space-time.

Since $\delta U^i/\delta u = 0$, we have $g_{ij} U^i U^j = \text{const.}$ along Γ , and we can write (1) in the form

$$\Omega(P'P) = \Omega(x', x) = \frac{1}{2}(u_1 - u_0)^2 g_{ij} U^i U^j, \quad (2)$$

with the last part evaluated anywhere on Γ . Further, we can choose u so that the end-values are $u_0 = 0$, $u_1 = 1$, and then we have

$$\Omega(P'P) = \Omega(x', x) = \frac{1}{2} g_{ij} U^i U^j, \quad (3)$$

evaluated anywhere on Γ . Also, as in I-(36), we may write

$$\Omega(P'P) = \Omega(x', x) = \frac{1}{2} \varepsilon L^2, \quad L = \int_{P'}^P ds. \quad (4)$$

Thus, to within the factor ε ($= \pm 1$), the world-function is half the square of the measure of the geodesic joining P' and P .

As stated above, we assume that P' and P have a unique geodesic

¹ This function was introduced into tensor calculus by RUSE [1931a, b]. Cf. also SYNGE [1931], YANO and MUTO [1936], SCHOUTEN [1954, p. 382]. It has been called the *distance function* and the *characteristic function*, but *world-function* seems the most appropriate title for it when it is used in relativity, because it determines the curved world of space-time.

passing through them. This will certainly be the case if the points are close enough to one another, but there are physical instances where it does not hold. Then the world-function is no longer single-valued, and the existence of partial derivatives must not be rashly assumed. A global theory of the world-function, covering such singular cases, would be very complicated, and we shall throughout assume that the geodesic $P'P$ is unique and the partial derivatives exist. With this understanding, the world-function Ω is a powerful tool for the execution of systematic approximations without abandoning the techniques of tensor calculus, as will be evident later in the book.

It is obvious from (4) that in flat space-time there exists a coordinate system such that

$$\begin{aligned}\Omega(x', x) &= \frac{1}{2}g_{ij}(x^{i'} - x^i)(x^{j'} - x^j), \\ g_{ij} &= \eta_{ij} = \text{diag}(1, 1, 1, -1).\end{aligned}\tag{5}$$

This is a useful formula to remember, because it suggests properties of Ω and its derivatives in the general case of curved space-time.

To understand the transformation properties of the world-function and its derivatives, it is best to think of two coordinate systems, say C' and C , in domains D' and D of space-time. These domains overlap, and in the overlap there is a smooth transformation $C \leftrightarrow C'$ (Fig. 1). (D' and D might very well both cover the whole of space-time, in which case the overlap is the whole of space-time.) The point P' lies

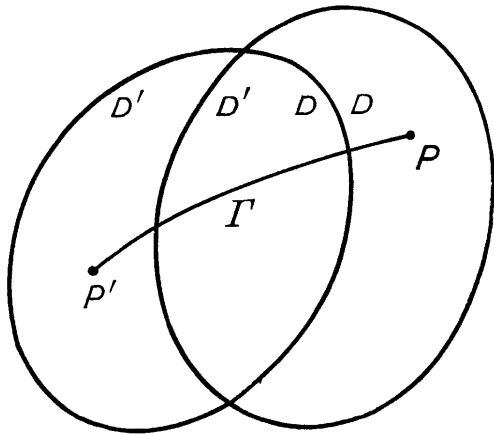


Fig. 1 – Overlapping domains D' , D with their own coordinate systems C' , C

in D' and has coordinates $x^{i'}$ in the system C' , while P lies in D and has coordinates x^i in the system C . The integral (1) is still meaningful if properly interpreted — we have to break the integral at some point in the overlap, and use for the two parts the coordinates C' and C respectively. The world-function $\Omega(x', x)$ is a *2-point invariant* in the sense that its value is unchanged if we transform *independently* the coordinate systems in D' and D . Briefly, we

may refer to such transformations as *transformations at P' and at P* .

Consider now the covariant derivatives of a 2-point invariant $I(x', x)$. All that follows holds in particular for $\Omega(x', x)$, but, since the

specific properties of Ω are not involved, it is clearer to speak of an arbitrary 2-point invariant I .

Covariant differentiation may be carried out with respect to the coordinates of P' or with respect to those of P . To avoid cumbersome notation, we shall denote these covariant derivatives by simple subscripts without the usual vertical stroke. The operations are applied in the order of the subscripts.

We have then before us such quantities as

$$I_{i'} = \frac{\partial I}{\partial x^{i'}}, \quad I_{i'j'} = \frac{\partial}{\partial x^{j'}} I_{i'} - \Gamma_{i'j'}^{a'} I_{a'}, \quad (6)$$

where the I 's are evaluated at P' , and

$$I_i = \frac{\partial I}{\partial x^i}, \quad I_{ij} = \frac{\partial}{\partial x^j} I_i - \Gamma_{ij}^a I_a, \quad (7)$$

where the I 's are evaluated at P . These quantities are of course functions of the coordinates of P' and P , and they are *2-point tensors*. It is clear that the quantities (6) are respectively a covariant vector and a covariant tensor of the second rank with respect to transformations at P' , but they are invariants with respect to transformations at P . There is no single simple term to describe these transformation properties, but they are at once evident from the notation. Similar remarks apply to the quantities (7) *mutatis mutandis*.

We have also such quantities as

$$I_{i'j} = \frac{\partial}{\partial x^j} I_{i'}, \quad I_{ij'} = \frac{\partial}{\partial x^{j'}} I_i, \quad (8)$$

$$I_{i'jk} = \frac{\partial}{\partial x^k} I_{i'j} - \Gamma_{jk}^a I_{i'a}. \quad (9)$$

Here the tensor properties are precisely as suggested by the notation. Each of (8) is a covariant vector under either transformation, while (9) is a covariant vector for transformations at P' and a covariant tensor of the second rank for transformations at P . The above illustrative examples should enable the reader to handle any covariant derivative.

We may raise subscripts by $g^{i'j'}$ at P' and by g^{ij} at P . Thus

$$I^{i'} = g^{i'j'} I_{j'}, \quad I^i = g^{ij} I_j. \quad (10)$$

It is obvious from (8) that

$$I_{ij'} = I_{j'i}. \quad (11)$$

This is a particular case covered by the following general *rule of interchange: the value of any covariant derivative is unaltered by interchange of primed and unprimed subscripts, provided the order of the primed subscripts and the order of the unprimed subscripts are separately preserved*. To prove this rule, let us write tentatively

$$I_{\dots ij'} = I_{\dots j'i}, \quad (12)$$

where the dots on the two sides stand for the same set of subscripts, primed or unprimed or both. This is certainly a tensor equation with respect to both transformations. We test its truth by using coordinate systems such that the I 's vanish at chosen positions of P' and P . Then, for the chosen P and any P' , we have

$$I_{\dots i} = \frac{\partial}{\partial x^i} I_{\dots}, \quad (13)$$

and hence, for P' and P both at their chosen positions,

$$I_{\dots ij'} = -\frac{\partial}{\partial x^{j'}} \frac{\partial}{\partial x^i} I_{\dots} \quad (14)$$

But likewise

$$I_{\dots j'i} = \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^{j'}} I_{\dots} \quad (15)$$

and since the expressions (14) and (15) are equal, (12) is verified for special coordinate systems. Therefore (12) is true in general, and it implies

$$I_{\dots ij' \dots} = I_{\dots j'i \dots}, \quad (16)$$

where the dots in corresponding positions stand for the same subscript, primed or unprimed. This means that, in any covariant derivative, it is permissible to interchange adjacent subscripts, provided one is primed and the other unprimed. From this the truth of the above rule of interchange is obvious.

Note that of course we cannot in general interchange two subscripts if both are primed or both unprimed [cf. the commutation rules of I-§ 5].

All the above formulae hold in particular for the world-function. By I-(41) we have

$$\Omega_{i'} = -(u_1 - u_0)U_{i'}, \quad \Omega_i = (u_1 - u_0)U_i, \quad (17)$$

where U^i is the tangent vector dx^i/du , and by 1-(42), if Γ is not null,

$$\Omega_{i'} = -L\lambda_{i'}, \quad \Omega_i = L\lambda_i, \quad (18)$$

where λ^i is the unit tangent vector to Γ . We recall that in (17) u is any special parameter, and in (18) we choose $du = ds$. These formulae are illustrated in Fig. 2.

By (17) we have

$$g^{ij}\Omega_i\Omega_j = (u_1 - u_0)^2 g^{ij}U_iU_j, \quad (19)$$

and so, by (2),

$$g^{ij}\Omega_i\Omega_j = 2\Omega. \quad (20)$$

Likewise

$$g^{i'j'}\Omega_{i'}\Omega_{j'} = 2\Omega. \quad (21)$$

In (20) and (21) we have the two partial differential equations satisfied by the world-function.

We note for reference the following obvious results:

$$\Omega_{i'j'} = \Omega_{j'i'}, \quad \Omega_{ij} = \Omega_{ji}, \quad (22)$$

$$\Omega_{i'j'...} = \Omega_{j'i'...}, \quad \Omega_{ij...} = \Omega_{ji...}, \quad (23)$$

where as usual dots in corresponding positions indicate the same subscript, primed or unprimed. This commutation holds only when the two subscripts are adjacent to the letter Ω .

§ 2. COINCIDENCE LIMITS

This section is devoted to the discussion of the limits of the covariant derivatives of the world-function when the two points P, P' tend to coincidence. We shall use a single coordinate system, so that $P \rightarrow P'$ implies $x^i \rightarrow x^{i'}$, with the following notation for these limits which we call *coincidence limits*:

$$\lim_{P \rightarrow P'} \Omega_{...} = [\Omega_{...}]. \quad (24)$$

These coincidence limits are useful only if they are independent of the path by which P tends to P' . To what extent they are in fact independent of path depends on the smoothness of the function g_{ij} , and a complete discussion of this difficult question is beyond the

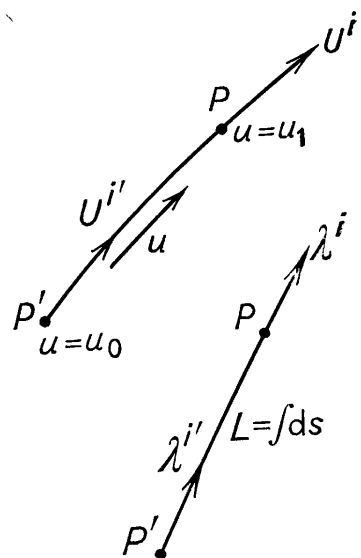


Fig. 2 – Illustrating the first derivatives of the world-function Ω

scope of this book. We shall however give a formal argument, assuming analyticity and the legitimacy of manipulations with infinite series.

The geodesic equation

$$\frac{\delta U^i}{\delta u} \equiv \frac{dU^i}{du} + \Gamma_{jk}^i U^j U^k = 0, \quad U^i = \frac{dx^i}{du}, \quad (25)$$

yields the power series

$$x^i = x^{i'} + u_1 U^{i'} - \frac{1}{2} u_1^2 \Gamma_{j'k'}^{i'} U^{j'} U^{k'} + \dots, \quad (26)$$

where $u = 0$ at P' and $u = u_1$ at P . Inversion gives

$$u_1 U^{i'} = \xi^i + \frac{1}{2} \Gamma_{j'k'}^{i'} \xi^j \xi^k + \dots, \quad \xi^i = x^i - x^{i'}. \quad (27)$$

Hence, by (2),

$$\begin{aligned} 2\Omega(x', x) &= u_1^2 g_{i'j'} U^{i'} U^{j'} \\ &= g_{i'j'} \xi^i \xi^j + A_{i'j'k'} \xi^i \xi^j \xi^k + \dots, \end{aligned} \quad (28)$$

the coefficients in this series being functions of $g_{i'j'}$ and its derivatives. Thus $\Omega(x', x)$ appears as an analytic function of its eight arguments, and the coincidence limits are consequently independent of path. Crude as this argument is mathematically, it shows that we are on the right track in selecting as world-function the function $\Omega(x', x)$ instead of, say, the geodesic measure of $P'P$. With that latter choice we would have encountered those indeterminacies and infinities which occur when, in Euclidean space, we differentiate the distance between two points with respect to their coordinates and proceed to coincidence limits.

To put the matter on an honest footing, let it be said that the following calculations depend on the assumptions that (a) the world-function $\Omega(x', x)$ is differentiable as often as required, and (b) the coincidence limits exist and are independent of the path by which P tends to P' .

The coincidence limits are of course functions of a single point, and it is a matter of indifference whether we call it P' or P . Notationally it is simpler to use P , and so, when a coincidence limit is evaluated as a tensor, the suffixes on that tensor will be written without primes. But it is essential to keep the primes inside the brackets $[\]$, unless sufficient reason is given for their deletion.

It is obvious from (1) and (17) that

$$[\Omega] = 0, \quad [\Omega_{i'}] = 0, \quad [\Omega_i] = 0, \quad (29)$$

and hence

$$[\Omega^{i'}] = 0, \quad [\Omega^i] = 0. \quad (30)$$

Differentiation of (20), which may be written

$$2\Omega = \dot{\Omega}_i \Omega^i, \quad (31)$$

gives

$$\Omega_j = \Omega_i \Omega_{.j}^i. \quad (32)$$

Let $u = 0$ at P' and $u = u_1$ at P . Multiply (32) by u_1 and use (17); this gives

$$U_j = U_i \Omega_{.j}^i. \quad (33)$$

Since the coincidence limit is to be independent of path, i.e. independent of the limit of U^i , we get

$$[\Omega_{.j}^i] = \delta_j^i, \quad [\Omega_{ij}] = g_{ij}. \quad (34)$$

To deal with the covariant derivatives of higher orders, we differentiate (32) again and again, obtaining

$$\Omega_{jk} = \Omega_{ik} \Omega_{.j}^i + \Omega_i \Omega_{.jk}^i, \quad (35)$$

$$\Omega_{jkm} = \Omega_{ikm} \Omega_{.j}^i + \Omega_{ik} \Omega_{.jm}^i + \Omega_{im} \Omega_{.jk}^i + \Omega_i \Omega_{.jkm}^i, \quad (36)$$

$$\begin{aligned} \Omega_{jkm p} = & \Omega_{ikm p} \Omega_{.j}^i + \Omega_{ikm} \Omega_{.jp}^i + \Omega_{ik p} \Omega_{.jm}^i + \Omega_{ik} \Omega_{.j m p}^i \\ & + \Omega_{im p} \Omega_{.jk}^i + \Omega_{im} \Omega_{.j k p}^i + \Omega_{ip} \Omega_{.j k m}^i + \Omega_i \Omega_{.j k m p}^i, \end{aligned} \quad (37)$$

and so on. We now go to the limit $P \rightarrow P'$, using the coincidence limits already evaluated. From (35) we get nothing, but (36) gives

$$[\Omega_{kjm}] + [\Omega_{mjk}] = 0. \quad (38)$$

But by (23) there is symmetry with respect to the first two subscripts, and this symmetry combined with the skew-symmetry in (38) leads easily to

$$[\Omega_{ijk}] = 0. \quad (39)$$

In view of the results already established, (37) gives in the limit

$$[\Omega_{kjm p}] + [\Omega_{mjk p}] + [\Omega_{pjk m}] = 0, \quad (40)$$

or, by (23),

$$[\Omega_{jkm p}] + [\Omega_{jmk p}] + [\Omega_{jpkm}] = 0. \quad (41)$$

By I-(96)

$$\Omega_{ijkm} - \Omega_{ijmk} = R^a_{ikm}\Omega_{aj} + R^a_{jkm}\Omega_{ia}, \quad (42)$$

and therefore

$$[\Omega_{jikm}] = [\Omega_{ijmk}], \quad (43)$$

so that these coincidence limits are symmetric in the last pair of subscripts as well as in the first pair; this may also be seen from (41).

By I-(94)

$$\Omega_{ijk} - \Omega_{ikj} = R^a_{ijk}\Omega_a. \quad (44)$$

Differentiating and going to the limit, we get

$$[\Omega_{ijkm}] - [\Omega_{ikjm}] = R_{mijk} = -R_{imjk}. \quad (45)$$

Interchange k and m , add, and use (43) twice:

$$2[\Omega_{ijkm}] - [\Omega_{ikjm}] - [\Omega_{imkj}] = -R_{imjk} - R_{ikjm}. \quad (46)$$

Hence by (41) and (43)

$$[\Omega_{ijkm}] = S_{ijkm}, \quad (47)$$

where S_{ijkm} is the *symmetrized Riemann tensor*, defined by

$$S_{ijkm} = -\frac{1}{3}(R_{ikjm} + R_{imjk}). \quad (48)$$

This tensor satisfies the symmetry equations

$$\begin{aligned} S_{ijkm} &= S_{jikm} = S_{ijmk} = S_{kmij}, \\ S_{iabc} + S_{ibca} + S_{icab} &= 0. \end{aligned} \quad (49)$$

It is as competent as the Riemann tensor to describe the curvature properties of space-time, having 20 independent components; there are 6 of the type ¹ S_{1122} , 12 of the type S_{1123} and 3 of the type S_{1234} , but these last satisfy the relation

$$S_{1234} + S_{1324} + S_{1423} = 0. \quad (50)$$

The R -tensor is given in terms of the S -tensor by

$$R_{ijkm} = -(S_{ikjm} - S_{imjk}) = -S_{ikmj} + S_{imkj}. \quad (51)$$

The coincidence limit $[\Omega_{ijkm}]$ satisfies, of course, the same symmetry equations as S_{ijkm} .

By extending the sequence of equations (35)–(37) by further differ-

¹ Note that $S_{1122} = -2S_{1212}$, $S_{1123} = -2S_{1213}$.

entiation, we can evaluate the coincidence limits for the derivatives of higher orders. Expressed in an umbral notation in which each numerical subscript stands for a letter, we have

$$[\Omega_{12345}] = -\frac{1}{4}(R_{13245} + R_{13254} + R_{14235} + R_{14253} + R_{15234} + R_{15243}), \quad (52)$$

the final subscripts on the R 's standing for covariant derivatives. The expression for $[\Omega_{123456}]$ is considerably more complicated ¹.

Since coincidence may be attained equivalently by letting $P \rightarrow P'$ or by letting $P' \rightarrow P$, it is clear that, in each of the above coincidence limits, we may put primes on all the subscripts, e.g.

$$[\Omega_{i'j'}] = g_{ij}, \quad [\Omega_{i'j'k'm'}] = S_{ijkm}. \quad (53)$$

The case where we have some subscripts primed and some unprimed must now be considered. For this we need the following lemma:

$$[\Omega_{...}]_k = [\Omega_{...k}] + [\Omega_{...k'}], \quad (54)$$

the dots standing for any set of subscripts, primed or unprimed, the same in each symbol.

To prove this lemma, we take a geodesic Γ and points P', P on it, the corresponding values of some special parameter of Γ being u', u , respectively. Consider the mixed covariant derivative

$$\Omega_{i_1 \dots i_p j_1' \dots j_q'} \quad (55)$$

Take a set of $(p + q)$ vectors, arbitrarily selected at any point of Γ and then subjected to parallel transport along Γ . Write $A^{i_1 \dots i_p}$ for the product of the first p vectors at P and $B^{j_1' \dots j_q'}$ for the product of the remaining q vectors at P' , and form the 2-point invariant

$$H(u', u) = \Omega_{i_1 \dots i_p j_1' \dots j_q'} A^{i_1 \dots i_p} B^{j_1' \dots j_q'}. \quad (56)$$

Taking $(u - u')$ small, so that P and P' are close together, and omitting terms of the second order, we have

$$\begin{aligned} H(u', u) &= H(u', u') + (u - u') \left(\frac{\partial H}{\partial u} \right)_{u=u'} \\ &= H(u', u') + (u - u') [\Omega_{...k}]_{P'} U^{k'} (A \cdots B \cdots)_{P'}, \end{aligned} \quad (57)$$

¹ Cf. SYNGE [1931]. The 4-index coincidence limits represent the so-called 'second extension' of the fundamental tensor, cf. VEBLEN [1927], p. 97.

where the coincidence limit is evaluated, as indicated, at P' , and $U^{k'}$ is the tangent vector ($dx^{k'}/d\mathfrak{u}$) at P' . For simplicity of writing, the suffixes of (56) have been replaced by dots in an obvious way. Likewise

$$\begin{aligned} H(\mathfrak{u}', \mathfrak{u}) &= H(\mathfrak{u}, \mathfrak{u}) + (\mathfrak{u}' - \mathfrak{u}) \left(\frac{\partial H}{\partial \mathfrak{u}'} \right)_{\mathfrak{u}'=\mathfrak{u}} \\ &= H(\mathfrak{u}, \mathfrak{u}) + (\mathfrak{u}' - \mathfrak{u}) [\Omega_{\dots k'}]_P U^k (A \dots B \dots)_P. \end{aligned} \quad (58)$$

Subtract (58) from (57), divide by $(\mathfrak{u}' - \mathfrak{u})$, and go to the limit $\mathfrak{u}' \rightarrow \mathfrak{u}$, thus bringing P' and P into coincidence. This gives

$$\frac{dH(\mathfrak{u}, \mathfrak{u})}{d\mathfrak{u}} = \{[\Omega_{\dots k}] + [\Omega_{\dots k'}]\} U^k A \dots B \dots, \quad (59)$$

everything being now evaluated at P . But

$$H(\mathfrak{u}, \mathfrak{u}) = [\Omega_{\dots}] A \dots B \dots, \quad (60)$$

and so

$$\frac{dH(\mathfrak{u}, \mathfrak{u})}{d\mathfrak{u}} = [\Omega_{\dots}]_k U^k A \dots B \dots. \quad (61)$$

The truth of (54) is now obvious if we compare (59) and (61) and remember that, at a given point P in space-time, we can select arbitrarily the direction of the geodesic Γ and the A - and B -vectors also.

It is convenient to rewrite (54) in the form

$$[\Omega_{\dots k'}] = [\Omega_{\dots}]_k - [\Omega_{\dots k}]. \quad (62)$$

When so written it becomes a machine for the manufacture of coincidence limits with primed subscripts, once those with unprimed subscripts are known. For example, if we take

$$\Omega_{\dots} = \Omega_i, \quad (63)$$

we get

$$[\Omega_{ik'}] = [\Omega_i]_k - [\Omega_{ik}]. \quad (64)$$

We already know that $[\Omega_i] = 0$ and hence $[\Omega_i]_k = 0$; also $[\Omega_{ik}] = g_{ik}$. Therefore

$$[\Omega_{ik'}] = -g_{ik}. \quad (65)$$

As a second example, take

$$\Omega_{\dots} = \Omega_{ij}. \quad (66)$$

Then (62) gives

$$[\Omega_{ijk'}] = [\Omega_{ij}]_k - [\Omega_{ijk}] \quad (67)$$

But $[\Omega_{ij}] = g_{ij}$ and so its covariant derivative vanishes; the 3-index symbol also vanishes, and we get

$$[\Omega_{ijk'}] = 0. \quad (68)$$

If we confine our attention to coincidence limits of order not greater than four (and these are the most important in the applications), we may use the following rules, which are easily verified:

- (i) Carry all primed subscripts to the right (by the general rule of interchange given in § 1).
- (ii) Delete the last prime and change sign.
- (iii) Carry this last subscript forward to join the other unprimed subscripts.
- (iv) Repeat the process until all the primes have disappeared.

For convenient reference, there follows a list of formulae of coincidence limits and of the symmetrized Riemann tensor:

$$\begin{aligned}
 [\Omega] &= 0, \quad [\Omega_i] = 0, \quad [\Omega_{i'}] = 0, \\
 [\Omega_{ij}] &= g_{ij}, \quad [\Omega_{ij'}] = [\Omega_{i'j}] = -g_{ij}, \quad [\Omega_{i'j'}] = g_{ij}, \\
 [\Omega_{ijk}] &= 0 \text{ (all expressions with three subscripts vanish),} \\
 [\Omega_{ijkm}] &= S_{ijkm}, \quad [\Omega_{ijkm'}] = -S_{ijkm}, \quad [\Omega_{ijk'm'}] = S_{ijkm}, \\
 [\Omega_{ij'k'm'}] &= -S_{imkj} = -S_{imjk}, \quad [\Omega_{i'j'k'm'}] = S_{ijkm}, \\
 S_{ijkm} &= -\frac{1}{3}(R_{ikjm} + R_{imjk}), \\
 R_{ijkm} &= -S_{ikjm} + S_{imjk} = -S_{ikmj} + S_{imkj}, \\
 S_{ijkm} &= S_{jikm} = S_{kmi j}, \\
 S_{iabc} + S_{ibca} + S_{icab} &= 0, \\
 (S_{iabjc} - S_{iacjb}) + (S_{ibcja} - S_{iba jc}) + (S_{icajb} - S_{icbj a}) &= 0.
 \end{aligned} \quad (69)$$

In the last formula, which is a consequence of the Bianchi identity I-(98), the fifth subscript indicates the covariant derivative.

§ 3. EVALUATION OF THE SECOND DERIVATIVES OF THE WORLD-FUNCTION BY USE OF THE PARALLEL PROPAGATOR

Approximations based on the neglect of small terms are very frequent in mathematical physics, and there is seldom any reason to object to them. One feels that if there is anything wrong, it will show up in some anomaly, and then one can revise the theory. Thus, in classical hydrodynamics, an approximation in which a fluid of *small*

viscosity is treated as having *no* viscosity leads to the paradox of d'Alembert (a body experiences no resistance when moving through water), and one turns to a more refined approximation in the theory of the boundary layer.

Once a theory has been given a clear mathematical formulation, one would like to develop it with mathematical precision. But usually that is neither possible (the physicist does not know enough mathematics) nor wise (details of mathematical precision obscure the general line of thought). This book aims at a middle course. Approximations are sometimes made naively — here is a small quantity, neglect its square and the square of its derivatives! At other times, the procedure is more critical. In particular, we must remember that *smallness* is a term which can be validly applied only to a dimensionless quantity, and even then it is only a matter of comparison. The force of attraction between the sun and the earth (to use Newtonian terminology) is dimensionless and its value¹ is about 3×10^{-22} . Is that small? It all depends on what we are comparing it with.

The work which follows is exact in the sense that uncalculated residues are collected under a symbol O_1 or O_2 , indicating orders of magnitude. The motivation lies in the fact that these terms are, in general, 'small'; it seems best to leave it to the reader to explore exceptional cases for himself. The theme of *smallness* will recur from time to time throughout the book, and may be diagnosed as the rumbling of a troubled mathematical conscience.

In the preceding section we evaluated the coincidence limits of the covariant derivatives of the world-function $\Omega(P'P)$. We now seek to evaluate those derivatives when P' and P are distinct points of space-time. The method permits of iteration, leading as close as we like to the true values, but we shall be most interested in approximate calculations for space-time in which the Riemann tensor is small (O_1), and we shall throw into a residue (O_2) terms quadratic in the Riemann tensor².

In this work we assume the existence of the derivatives of Ω , which means that *conjugate points* are not considered. Such points correspond to foci in the optics of rays emanating from a point-source, and it is easy to see that they exist in the physical applications of the theory (to planetary orbits, for example). Their existence, however, does not

¹ Cf. Appendix B.

² For an alternative procedure, see VII-§ 9.

invalidate the theory, provided we understand that P' and P are not conjugate. The whole question of conjugate points in Riemannian space-time is interesting, both mathematically and physically, but we shall not attempt to discuss it ¹.

As a preliminary step, we consider the equation of parallel transport, which reads, for any curve and any parametrization on it,

$$\delta\lambda^i/\delta u = 0. \quad (70)$$

Let the curve be a geodesic joining points P' and P . Then, given these points, the vector $\lambda^{i'}$ at P' determines by parallel transport the vector λ^i at P . From the linear homogeneous character of the differential equations (70), we know that λ^i are linear homogeneous functions of $\lambda^{i'}$, so that we may write

$$\lambda_i = g_{ij'}\lambda^{j'}. \quad (71)$$

Here the coefficient $g_{ij'}$ is independent of the vector chosen. It is determined ² by the points P' , P , and is a 2-point tensor; in fact it is a covariant vector with respect to each point. We call $g_{ij'}$ the *parallel propagator*. The notation may seem confusing, since g_{ij} and $g_{i'j'}$ are already in use for the metric tensor at P and P' . However, there need be no confusion, and, as we shall see later, the notation fits in nicely. If we let P' tend to P , we get the coincidence limit

$$[g_{ij'}] = g_{ij}. \quad (72)$$

There is no difficulty in raising subscripts:

$$\begin{aligned} g^i_{j'} &= g^{ik}g_{kj'}, & g^{j'}_i &= g_{ik'}g^{k'j'}, \\ g^{ij'} &= g^{ik}g_{km'}g^{m'j'}. \end{aligned} \quad (73)$$

If $\lambda^i_{(a)}$ is an orthonormal tetrad (OT) carried by parallel transport along the geodesic $P'P$, we have, by (71),

$$\lambda_{(a)i} = g_{ij'}\lambda^{j'}_{(a)}, \quad (74)$$

and so, by 1-(50) and 1-(48),

$$g_{ij'} = \lambda_{(a)i}\lambda^{(a)}_{j'} = \eta_{(ba)}\lambda^{(b)}_i\lambda^{(a)}_{j'} = g_{j'i}. \quad (75)$$

Thus the parallel propagator is symmetric.

We might have defined the parallel propagator by (75), but then it

¹ For some remarks on conjugate points, see SYNGE [1926a]. See also § 9.

² We assume that there is a unique geodesic joining P' to P .

might not have been obvious that $g_{ij'}$ is really a 2-point tensor independent of the OT. However, this independence is easily verified by submitting the OT to a Lorentz transformation 1-(52).

We note in passing that Fermi-Walker transport (1-§ 4) also has a propagator, say $W_{ij'}$, such that we have for F-W transport along any curve (other than a null curve)

$$\lambda_i = W_{ij'} \lambda^{j'}, \quad (76)$$

with $W_{ij'}$ given by

$$W_{ij'} = \lambda_{(a)i} \lambda_{j'}^{(a)} = W_{j'i}. \quad (77)$$

However, this is not a true 2-point tensor, since it depends on the curve joining P' and P . In the case of $g_{ij'}$ we effectively eliminated the curve by making it a geodesic, and if we do that for F-W transport we get $W_{ij'} = g_{ij'}$.

In order to evaluate the covariant derivatives of the world-function Ω , we return to the equation of geodesic deviation 1-(130):

$$\frac{\delta^2 V^i}{\delta u^2} + K^i_{.m} V^m = 0, \quad K_{im} = R_{ipmq} U^p U^q. \quad (78)$$

In 1-§ 6, we discussed the solution in terms of invariants; now we shall use the parallel propagator.

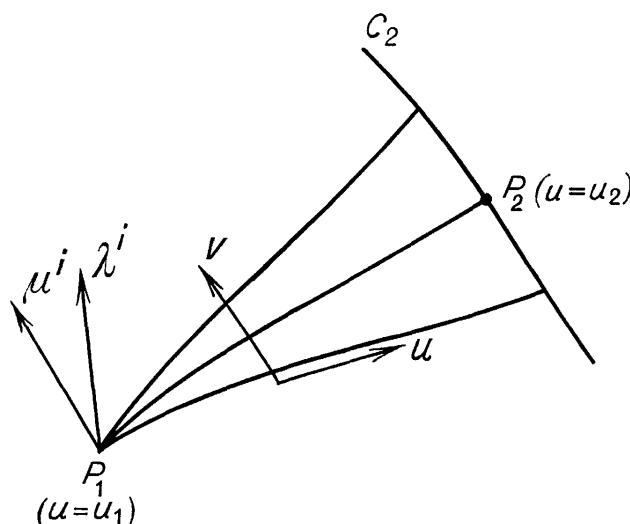


Fig. 3 – Calculation of the covariant derivatives of the world-function Ω

Fig. 3 shows a family of geodesics emanating from a point P_1 and intersecting a curve C_2 with equations $x^i = x^i(v)$, on which P_2 is any point. Choosing a special parameter u on these geodesics, with fixed end values $u = u_1$ at P_1 and $u = u_2$ on C_2 , we have a 2-space

$x^i = x^i(u, v)$. Writing, as in I-§ 6,

$$U^i = \partial x^i / \partial u, \quad V^i = \partial x^i / \partial v, \quad (79)$$

we have, all over the 2-space

$$\frac{\delta U^i}{\delta v} = \frac{\delta V^i}{\delta u}, \quad \frac{\delta U^i}{\delta u} = 0, \quad (80)$$

and, since P_1 is fixed,

$$V^{i_1} = 0, \quad (81)$$

the secondary (numerical) suffix indicating that the vector is to be taken at P_1 .

The situation is precisely as considered in I-§ 6 except that, in order to keep the complexity of the calculations under control, we have fixed P_1 . We have the deviation equation (78). Let λ^i be a vector chosen arbitrarily at P_1 and carried by parallel transport along the geodesics, so that

$$\delta \lambda^i / \delta u = 0. \quad (82)$$

Multiply (78) by $G(u, u') \lambda_i du$, where $G(u, u')$ is the symmetric Green's function I-(144), and integrate from P_1 to P_2 . This gives

$$\int_{u_1}^{u_2} G D^2(\lambda_i V^i) du + \int_{u_1}^{u_2} G K_{im} \lambda^i V^m du = 0, \quad (83)$$

where $D = d/du$, and hence, as in I-(147),

$$\int_{u_1}^{u_2} D G D(\lambda_i V^i) du = \int_{u_1}^{u_2} G K_{km} \lambda^k V^m du. \quad (84)$$

Further, as in I-(149) but remembering (81), we get

$$\lambda_{i'} V^{i'} = k(u' - u_1) \lambda_{j_2} V^{j_2} + \int_{u_1}^{u_2} G K_{km} \lambda^k V^m du, \quad (85)$$

where $k^{-1} = u_2 - u_1$. Differentiation with respect to u' gives

$$\begin{aligned} \lambda_{i'} \frac{\delta V^{i'}}{\delta u'} &= k \lambda_{j_2} V^{j_2} - k \int_{u_1}^{u'} (u - u_1) K_{km} \lambda^k V^m du \\ &\quad + k \int_{u'}^{u_2} (u_2 - u) K_{km} \lambda^k V^m du. \end{aligned} \quad (86)$$

Since the vector λ^i may be chosen arbitrarily at any point of $P_1 P_2$, say at $u = u'$, we get from (85) and (86), on introducing the parallel

propagator,

$$V_{i'} = k(u' - u_1)g_{i'j_2}V^{j_2} + \int_{u_1}^{u_2} GK^{km}g_{i'k}V_m du, \quad (87)$$

$$\begin{aligned} \frac{\delta U_{i'}}{\delta v} = \frac{\delta V_{i'}}{\delta u'} &= kg_{i'j_2}V^{j_2} - k \int_{u_1}^{u'} (u - u_1)K^{km}g_{i'k}V_m du \\ &\quad + k \int_{u'}^{u_2} (u_2 - u)K^{km}g_{i'k}V_m du. \end{aligned} \quad (88)$$

These formulae are accurate. By iteration, we can obtain ¹ from (87) the deviation vector to any desired degree of accuracy, and its absolute derivative is then given by (88). In this iteration process, the successive terms proceed in powers of the Riemann tensor. If this tensor is small (O_1), we have (delete the primes)

$$V_i = k(u - u_1)g_{ij_2}V^{j_2} + O_1, \quad (89)$$

$$\frac{\delta U_i}{\delta v} = \frac{\delta V_i}{\delta u} = kg_{ij_2}V^{j_2} + O_1. \quad (90)$$

These crude approximations are essential in later work. But we have also, more accurately ², by (88) evaluated at P_1 and P_2 ,

$$\frac{\delta U_{i_1}}{\delta v} = kg_{i_1j_2}V^{j_2} + k^2 \int_{u_1}^{u_2} (u_2 - u)(u - u_1)K^{km}g_{i_1k}g_{j_2m}V^{j_2} du + O_2, \quad (91)$$

$$\frac{\delta U_{i_2}}{\delta v} = kg_{i_2j_2}V^{j_2} - k^2 \int_{u_1}^{u_2} (u - u_1)^2 K^{km}g_{i_2k}g_{j_2m}V^{j_2} du + O_2. \quad (92)$$

¹ If the geodesics from P_1 meet again at P_2 , then $V^{j_2} = 0$. The integral equation (87) becomes homogeneous and the method breaks down completely. This is the excluded case of conjugate points.

² In (91) and subsequent formulae the symbol O_2 stand for an integral with an integrand quadratic in the Riemann tensor. For example, in (91) O_2 stands for

$$k \int_{u_1}^{u_2} \int_{u_1}^{u_2} (u_2 - u)G(u, u')K^{km}K^{a'b'}g_{i_1k}g_{a'm}V_{b'} du du'. \quad (91a)$$

In regard to smallness, we think primarily of the Riemann tensor as small and the range $(u_2 - u_1)$ as finite, and then the symbol O_2 indicates second-order smallness, as it should; but if the range is large, the question of smallness needs careful examination. Perhaps it is well to emphasize that we are not basing smallness on smallness of the range.

The preceding calculations do not involve the world-function. We now write, as in (17),

$$\Omega_{i_1} = - (u_2 - u_1)U_{i_1}, \quad \Omega_{i_2} = (u_2 - u_1)U_{i_2}. \quad (93)$$

Carrying P_2 along C_2 , and differentiating with respect to v , we get (since P_1 does not move)

$$\Omega_{i_1 j_2} V^{j_2} = - k^{-1} \frac{\delta U_{i_1}}{\delta v}, \quad \Omega_{i_2 j_2} V^{j_2} = k^{-1} \frac{\delta U_{i_2}}{\delta v}. \quad (94)$$

For the terms on the right, we have expressions as in (91) and (92). Now, given P_1 and P_2 , the curve C_2 may be drawn arbitrarily through P_2 , so that V^{j_2} is arbitrary and we can cancel it out. Thus we get expressions for $\Omega_{i_1 j_2}$ and $\Omega_{i_2 j_2}$. Then we can interchange the numbers 1 and 2 throughout, remembering to change the sign of k . Further, we have (if we want to use it) the general rule of interchange (§ 1). Finally, introducing the S-tensor from § 2, we get the following expressions for the *second-order covariant derivatives of the world-function*:

$$\begin{aligned} \Omega_{i_1 j_1} &= g_{i_1 j_1} + \frac{3}{2}k \int_{u_1}^{u_2} (u_2 - u)^2 g_{i_1 a} g_{j_1 b} S^{abpq} U_p U_q du + O_2, \\ \Omega_{i_1 j_2} &= \Omega_{j_2 i_1} = - g_{i_1 j_2} \\ &\quad + \frac{3}{2}k \int_{u_1}^{u_2} (u_2 - u)(u - u_1) g_{i_1 a} g_{j_2 b} S^{abpq} U_p U_q du + O_2, \quad (95) \\ \Omega_{i_2 j_2} &= g_{i_2 j_2} + \frac{3}{2}k \int_{u_1}^{u_2} (u - u_1)^2 g_{i_2 a} g_{j_2 b} S^{abpq} U_p U_q du + O_2, \\ k^{-1} &= u_2 - u_1, \quad S^{abpq} = - \frac{1}{3}(R^{apbq} + R^{aqbp}). \end{aligned}$$

These formulae may be written a little more neatly in terms of invariant components, as in I-(54), on an OT $\lambda_{(a)}^i$ which is carried by parallel transport along the geodesic $P_1 P_2$. If we multiply the first of (95) by $\lambda_{(m)}^{i_1} \lambda_{(n)}^{j_1}$, this product passes under the sign of integration, and we get

$$\begin{aligned} g_{i_1 a} \lambda_{(m)}^{i_1} &= \lambda_{(m)a}, \quad g_{j_1 b} \lambda_{(n)}^{j_1} = \lambda_{(n)b}, \\ \lambda_{(m)a} \lambda_{(n)b} S^{abpq} U_p U_q &= S_{(mnr s)} U^{(r)} U^{(s)}. \end{aligned} \quad (96)$$

Treating the other equations similarly, and noting that $U^{(r)}$ is constant along the geodesic, we get the following expressions for the invariant

components, with $\eta_{(mn)} = \text{diag}(1, 1, 1, -1)$ ¹:

$$\begin{aligned}\Omega_{(m_1 n_1)} &= \eta_{(mn)} + \frac{3}{2}kU^{(r)}U^{(s)} \int_{u_1}^{u_2} (u_2 - u)^2 S_{(mnr s)} du + O_2, \\ \Omega_{(m_1 n_2)} &= -\eta_{(mn)} + \frac{3}{2}kU^{(r)}U^{(s)} \int_{u_1}^{u_2} (u_2 - u)(u - u_1) S_{(mnr s)} du + O_2, \\ \Omega_{(m_2 n_2)} &= \eta_{(mn)} + \frac{3}{2}kU^{(r)}U^{(s)} \int_{u_1}^{u_2} (u - u_1)^2 S_{(mnr s)} du + O_2.\end{aligned}\quad (97)$$

§ 4. EVALUATION OF THE COVARIANT DERIVATIVES OF THE PARALLEL PROPAGATOR

In later work, we need to use the fact that the covariant derivatives of the parallel propagator $g_{ij'}$ are small in a space-time of small curvature. That fact is really obvious intuitively, for, in a flat space-time, it is easy to see that these covariant derivatives vanish. However, the parallel propagator is an essential element of space-time, and we shall devote this section to an evaluation of its covariant derivatives.

We work with Fig. 3. Let λ^i and μ^i be vectors chosen arbitrarily at P_1 and carried by parallel transport along the geodesics. Over the 2-space of geodesics, we have then

$$\delta\lambda^i/\delta u = 0, \quad \delta\mu^i/\delta u = 0, \quad (98)$$

and also, since P_1 is fixed,

$$\delta\lambda^{i_1}/\delta v = 0, \quad \delta\mu^{i_1}/\delta v = 0. \quad (99)$$

We have

$$\lambda_{i_2} = g_{i_2 j_1} \lambda^{j_1}, \quad (100)$$

and hence, on differentiation with respect to v ,

$$\frac{\delta\lambda_{i_2}}{\delta v} = g_{i_2 j_1 k_2} \lambda^{j_1} V^{k_2}, \quad (101)$$

where the third subscript on the g -term indicates a covariant derivative (in this case with respect to P_2). But, by the commutation rule I-(95) combined with (98), we have

$$\frac{\delta}{\delta u} \frac{\delta\lambda_i}{\delta v} = R_{ijmn} \lambda^j U^m V^n, \quad (102)$$

¹ To make the notation clear, note that, in the first or second line, m_1 on the left takes the same numerical value as m on the right; the secondary (numerical) suffix indicates the point P_1 .

and hence

$$\frac{d}{du} \left(\mu^i \frac{\delta \lambda_i}{\delta v} \right) = R_{abmn} \mu^a \lambda^b U^m V^n. \quad (103)$$

Integrating and using (99), we get

$$\begin{aligned} \mu^{i_2} \frac{\delta \lambda_{i_2}}{\delta v} &= \int_{u_1}^{u_2} R^{abmn} \mu_a \lambda_b U_m V_n du \\ &= \mu^{i_2} \lambda^{j_1} \int_{u_1}^{u_2} g_{i_2 a} g_{j_1 b} R^{abmn} U_m V_n du. \end{aligned} \quad (104)$$

Since μ^{i_2} may be regarded as arbitrary, we simply cross it out. Further, we may substitute from (101) and treat λ^{j_1} as arbitrary, so that it too may be crossed out. This gives

$$g_{i_2 j_1 k_2} V^{k_2} = - \int_{u_1}^{u_2} g_{i_2 a} g_{j_1 b} R^{abmn} V_m U_n du. \quad (105)$$

So far the work is exact. But now we substitute for the deviation vector V_m in the integral from (89), and get (treating V^{k_2} as arbitrary)

$$g_{i_2 j_1 k_2} = -k \int_{u_1}^{u_2} (u - u_1) g_{i_2 a} g_{j_1 b} g_{k_2 c} R^{abcq} U_q du + O_2. \quad (106)$$

This formula contains all the first-order covariant derivatives of the parallel propagator, in view of its symmetry and the possibility of interchanging the numbers 1 and 2. Thus we have ¹

$$\begin{aligned} g_{i_1 j_2 k_1} &= k \int_{u_1}^{u_2} (u_2 - u) g_{i_1 a} g_{j_2 b} g_{k_1 c} R^{abcq} U_q du + O_2, \\ g_{i_1 j_2 k_2} &= k \int_{u_1}^{u_2} (u - u_1) g_{i_1 a} g_{j_2 b} g_{k_2 c} R^{abcq} U_q du + O_2, \end{aligned} \quad (107)$$

$$k^{-1} = u_2 - u_1.$$

If we introduce an OT $\lambda_{(a)}^i$ with parallel transport along the geodesic

¹ First, interchange 1 and 2 in (106), remembering to interchange u_1 and u_2 and hence to change k into $-k$. Secondly, interchange i and j in (106) and use the symmetry of the propagator and the skew-symmetry of the Riemann tensor.

P_1P_2 and multiply the first of (107) by $\lambda_{(r)}^{i_1}\lambda_{(s)}^{j_2}\lambda_{(t)}^{k_1}$, the equation takes a simpler form. Treating the second of (107) similarly, we get, in terms of components on the OT,

$$\begin{aligned} g_{(a_1b_2c_1)} &= kU^{(q)} \int_{u_1}^{u_2} (u_2 - u) R_{(abcq)} du + O_2, \\ g_{(a_1b_2c_2)} &= kU^{(q)} \int_{u_1}^{u_2} (u - u_1) R_{(abcq)} du + O_2. \end{aligned} \quad (108)$$

It is now easy to compute, to this order of approximation, the co-variant derivatives of higher orders. To get those of the fourth order, we work with Fig. 3 and differentiate (107) with respect to v . Since the expressions in (107) are O_1 , it is unnecessary to touch the propagators under the sign of integration, since their derivatives would give O_2 terms. But we have to differentiate the Riemann tensor and U_q . In this procedure, (89) and (90) are the key formulae. Thus ¹

$$\begin{aligned} \frac{\delta}{\delta v} R_{abcq} &= R_{abcq;d} V_d = R_{abcq;d} k(u - u_1) g_{dm_2} V^{m_2} + O_2, \\ \frac{\delta}{\delta v} U_q &= k g_{qm_2} V^{m_2} + O_1. \end{aligned} \quad (109)$$

In this way we obtain the following formulae in terms of components on a parallel-transported OT:

$$\begin{aligned} g_{(a_1b_2c_1d_1)} &= -k^2 \int_{u_1}^{u_2} (u_2 - u) R_{(abcd)} du \\ &\quad + k^2 U^{(q)} \int_{u_1}^{u_2} (u_2 - u)^2 R_{(abcq;d)} du + O_2, \\ g_{(a_1b_2c_1d_2)} &= k^2 \int_{u_1}^{u_2} (u_2 - u) R_{(abcd)} du \\ &\quad + k^2 U^{(q)} \int_{u_1}^{u_2} (u_2 - u)(u - u_1) R_{(abcq;d)} du + O_2, \\ g_{(a_1b_2c_2d_2)} &= k^2 \int_{u_1}^{u_2} (u - u_1) R_{(abcd)} du + k^2 U^{(q)} \int_{u_1}^{u_2} (u - u_1)^2 R_{(abcq;d)} du + O_2. \end{aligned} \quad (110)$$

Changing the notation slightly, so that the two points are P' and P instead of P_1 and P_2 , we get the following coincidence limits when P'

¹ The fifth suffix on R indicates the covariant derivative, with the subscript raised in the usual manner.

tends to P :

$$\begin{aligned} [g_{ij'}] &= g_{ij}, & [g_{ij'k}] &= 0, & [g_{ij'k'}] &= 0, \\ [g_{ij'km}] &= -\frac{1}{2}R_{ijkm}, & [g_{ij'km'}] &= \frac{1}{2}R_{ijkm}. \end{aligned} \quad (111)$$

These last formulae have a close connection with the parallel transport of a vector round a small circuit.

§ 5. EVALUATION OF THE HIGHER DERIVATIVES OF THE WORLD-FUNCTION

To evaluate the higher covariant derivatives of the world-function Ω , we start from the first and third of (95):

$$\begin{aligned} \Omega_{i_1 j_1} &= g_{i_1 j_1} + \frac{3}{2}k \int_{u_1}^{u_2} (u_2 - u)^2 g_{i_1 a} g_{j_1 b} S^{abpq} U_p U_q du + O_2, \\ \Omega_{i_2 j_2} &= g_{i_2 j_2} + \frac{3}{2}k \int_{u_1}^{u_2} (u - u_1)^2 g_{i_2 a} g_{j_2 b} S^{abpq} U_p U_q du + O_2. \end{aligned} \quad (112)$$

Our plan is to use the scheme shown in Fig. 3 and differentiate with respect to v . In view of (107), we can (to the order of approximation required) leave the propagators in the integrals untouched. As for the other terms, we get zero from the leading terms on the right hand sides of (112); for $g_{i_1 j_1}$ is independent of P_2 and the covariant derivative of $g_{i_2 j_2}$ vanishes. Further, we, have, as in (109),

$$\begin{aligned} \frac{\delta}{\delta v} U_p &= k g_{pk_2} V^{k_2} + O_1, \\ \frac{\delta}{\delta v} S^{abpq} &= S^{abpq} k (u - u_1) g_{ck_2} V^{k_2} + O_2, \end{aligned} \quad (113)$$

where the fifth suffix on S indicates covariant differentiation.

By straightforward calculation, interchange of the numbers 1 and 2, interchange of suffixes, and the general rule of interchange (§ 1), we easily find all the third-order derivatives of the world-function. Thus, from the first of (112) we get

$$\begin{aligned} \Omega_{i_1 j_1 k_2} &= 3k^2 \int_{u_1}^{u_2} (u_2 - u)^2 g_{i_1 a} g_{j_1 b} g_{k_2 c} S^{abcq} U_q du \\ &+ \frac{3}{2}k^2 \int_{u_1}^{u_2} (u_2 - u)^2 (u - u_1) g_{i_1 a} g_{j_1 b} g_{k_2 c} S^{abpq} U_p U_q du + O_2. \end{aligned} \quad (114)$$

This, and other similar formulae, are best expressed in terms of components on an OT parallel-transported along P_1P_2 . We find the following values for the *invariant components of the third-order covariant derivatives of the world-function* $\Omega(P_1P_2)$:

$$\begin{aligned}
 \Omega_{(a_1b_1c_1)} &= -3k^2U^{(q)} \int_{u_1}^{u_2} (u_2 - u)^2 S_{(abcq)} du \\
 &\quad + \frac{3}{2}k^2U^{(p)}U^{(q)} \int_{u_1}^{u_2} (u_2 - u)^3 S_{(abpqe)} du + O_2, \\
 \Omega_{(a_1b_1c_2)} &= 3k^2U^{(q)} \int_{u_1}^{u_2} (u_2 - u)^2 S_{(abcq)} du \\
 &\quad + \frac{3}{2}k^2U^{(p)}U^{(q)} \int_{u_1}^{u_2} (u_2 - u)^2 (u - u_1) S_{(abpqe)} du + O_2, \\
 \Omega_{(a_1b_2c_2)} &= -3k^2U^{(q)} \int_{u_1}^{u_2} (u - u_1)^2 S_{(bcaq)} du \\
 &\quad + \frac{3}{2}k^2U^{(p)}U^{(q)} \int_{u_1}^{u_2} (u_2 - u)(u - u_1)^2 S_{(bcpqa)} du + O_2, \\
 \Omega_{(a_2b_2c_2)} &= 3k^2U^{(q)} \int_{u_1}^{u_2} (u - u_1)^2 S_{(abcq)} du \\
 &\quad + \frac{3}{2}k^2U^{(p)}U^{(q)} \int_{u_1}^{u_2} (u - u_1)^3 S_{(abpqe)} du + O_2.
 \end{aligned} \tag{115}$$

Note that in all these formulae *except the third* the suffixes abc on the S -term occur in their alphabetical order. In view of the general rule of interchange (§ 1), *all* the third-order derivatives are contained in the above list. To pass from invariant to tensor form, we may put the U -terms into the integrals and substitute as follows:

$$\begin{aligned}
 U^{(q)} &= U^i \lambda_i^{(q)}, & S_{(abcq)} &= S_{ijkl} \lambda_{(a)}^i \lambda_{(b)}^j \lambda_{(c)}^k \lambda_{(d)}^m, \\
 \lambda_{(a)i} &= g_{ij_1} \lambda_{(a)}^{j_1} = g_{ij_2} \lambda_{(a)}^{j_2}, & \Omega_{(a_1b_1c_2)} &= \Omega_{i_1j_1k_2} \lambda_{(a)}^{i_1} \lambda_{(b)}^{j_1} \lambda_{(c)}^{k_2},
 \end{aligned} \tag{116}$$

and similar expressions.

We used in (112) only the first and third of (95); it is an interesting exercise to obtain $\Omega_{(a_1b_2c_2)}$ from the second of (95) and verify that the result is the same as that written in (115).

To get the covariant derivatives of the fourth order, we have merely to repeat the above process. We get the following formulae for the *invariant components of the fourth-order covariant derivatives of the*

world-function $\Omega(P_1 P_2)$:

$$\begin{aligned}
\Omega_{(a_1 b_1 c_1 d_1)} &= 3k^3 \int_{u_1}^{u_2} (u_2 - u)^2 S_{(abcd)} du \\
&\quad - 3k^3 U^{(q)} \int_{u_1}^{u_2} (u_2 - u)^3 (S_{(abcq d)} + S_{(abdqc)}) du \\
&\quad + \frac{3}{2} k^3 U^{(p)} U^{(q)} \int_{u_1}^{u_2} (u_2 - u)^4 S_{(abpqc d)} du + O_2, \\
\Omega_{(a_1 b_1 c_1 d_2)} &= - 3k^3 \int_{u_1}^{u_2} (u_2 - u)^2 S_{(abcd)} du \\
&\quad - 3k^3 U^{(q)} \int_{u_1}^{u_2} (u_2 - u)^2 (u - u_1) S_{(abcq d)} du \\
&\quad + 3k^3 U^{(q)} \int_{u_1}^{u_2} (u_2 - u)^3 S_{(abdqc)} du \\
&\quad + \frac{3}{2} k^3 U^{(p)} U^{(q)} \int_{u_1}^{u_2} (u_2 - u)^3 (u - u_1) S_{(abpqc d)} du + O_2, \\
\Omega_{(a_1 b_1 c_2 d_2)} &= 3k^3 \int_{u_1}^{u_2} (u_2 - u)^2 S_{(abcd)} du \\
&\quad + 3k^3 U^{(q)} \int_{u_1}^{u_2} (u_2 - u)^2 (u - u_1) (S_{(abcq d)} + S_{(abdqc)}) du \\
&\quad + \frac{3}{2} k^3 U^{(p)} U^{(q)} \int_{u_1}^{u_2} (u_2 - u)^2 (u - u_1)^2 S_{(abpqc d)} du + O_2.
\end{aligned} \tag{117}$$

That is enough to write down, because the others can be obtained by interchanges. The above expressions can be altered in form by integrations by parts, but without any great advantage. It is of interest to note that by the general rule of interchange (§ 1), we know that

$$\Omega_{(a_1 b_1 c_2 d_2)} = \Omega_{(c_2 d_2 a_1 b_1)}. \tag{118}$$

This means that the last expression in (117) must be invariant in form under the interchanges

$$1 \leftrightarrow 2, \quad a \leftrightarrow c, \quad b \leftrightarrow d. \tag{119}$$

It is by no means obvious from inspection that this is so; the symmetries of the S-tensor are involved.

§ 6. SOLUTION OF FINITE GEODESIC TRIANGLES IN SPACE-TIME OF SMALL CURVATURE

Just as the survey of Euclidean space is based on three-dimensional triangulation and the solution of triangles with straight sides, so the astronomical survey of space-time calls for four-dimensional triangulation and the solution of geodesic triangles. Some of the sides of these triangles may be null geodesics; the following argument covers them, the word 'geodesic' being understood to include 'null geodesic'. Indeed, when we use the method of the world-function, it is seldom necessary to treat null geodesics separately.

In the solution of geodesic triangles, there are two cases for which significant results can be obtained without too much labour. These are (i) the case where the Riemann tensor is small, and (ii) the case where the triangle is small ¹. We shall now proceed with the solution of a finite geodesic triangle, assuming the Riemann tensor small, an assumption made in the preceding calculations.

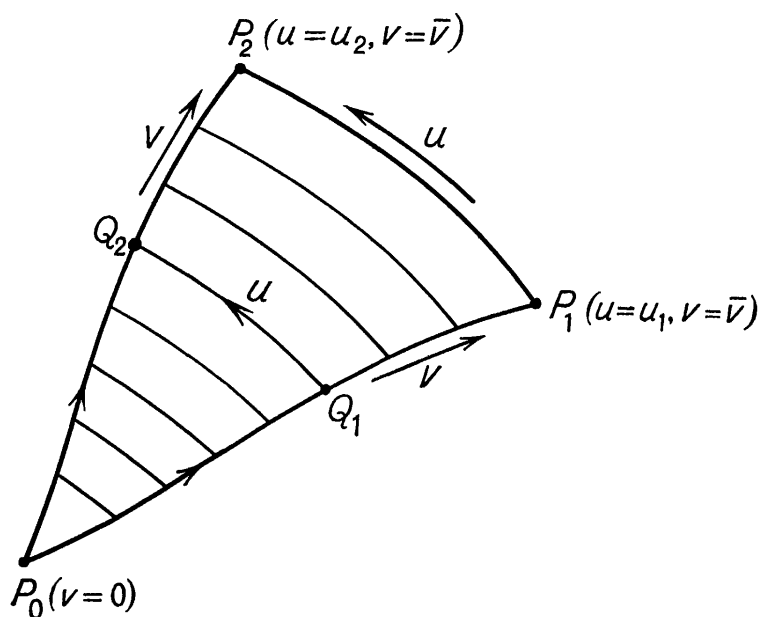


Fig. 4 – Solution of finite geodesic triangle

Consider the geodesic triangle $P_0P_1P_2$ (Fig. 4). Let v be a special parameter on P_0P_1 and P_0P_2 , running between the same terminal values on both geodesics, $v = 0$ at P_0 and $v = \bar{v}$ at P_1 and P_2 . Let Q_1 and Q_2 be current points on the two geodesics having the same value of v , and let the geodesic Q_1Q_2 be drawn. All such geodesics (the family of course includes P_1P_2) form a 2-space. Let u be a special parameter

¹ See remarks on smallness in § 3.

on each of these cross-geodesics, running from $u = u_1$ on P_0P_1 and $u = u_2$ on P_0P_2 . We have then a 2-space $x^i = x^i(u, v)$, all the parametric lines of u being geodesics, and two of the parametric lines of v being geodesics, viz. $u = u_1$ and $u = u_2$. Writing as usual $U^i = \partial x^i / \partial u$, $V^i = \partial x^i / \partial v$, we have

$$\frac{\delta U^i}{\delta v} = \frac{\delta V^i}{\delta u}, \quad \frac{\delta U^i}{\delta u} = 0, \quad (120)$$

$$U^i = 0 \text{ for } v = 0, \quad \frac{\delta V^i}{\delta v} = 0 \text{ for } u = u_1 \text{ and } u = u_2.$$

The world-function of Q_1 and Q_2 is a function of v only, and we write $\Omega(Q_1Q_2) = \Omega(v)$. Then

$$D_v \Omega = \Omega_{i_1} V^{i_1} + \Omega_{i_2} V^{i_2}, \quad (121)$$

where $D_v = d/dv$ and the secondary (numerical) subscripts refer to Q_1 and Q_2 . Further, by (120),

$$D_v^2 \Omega = \Omega_{i_1 j_1} V^{i_1} V^{j_1} + 2\Omega_{i_1 j_2} V^{i_1} V^{j_2} + \Omega_{i_2 j_2} V^{i_2} V^{j_2}, \quad (122)$$

$$D_v^3 \Omega = \Omega_{i_1 j_1 k_1} V^{i_1} V^{j_1} V^{k_1} + 3\Omega_{i_1 j_1 k_2} V^{i_1} V^{j_1} V^{k_2} \\ + 3\Omega_{i_1 j_2 k_2} V^{i_1} V^{j_2} V^{k_2} + \Omega_{i_2 j_2 k_2} V^{i_2} V^{j_2} V^{k_2}. \quad (123)$$

$$D_v^4 \Omega = \Omega_{i_1 j_1 k_1 m_1} V^{i_1} V^{j_1} V^{k_1} V^{m_1} + 4\Omega_{i_1 j_1 k_1 m_2} V^{i_1} V^{j_1} V^{k_1} V^{m_2} \\ + 6\Omega_{i_1 j_1 k_2 m_2} V^{i_1} V^{j_1} V^{k_2} V^{m_2} + 4\Omega_{i_1 j_2 k_2 m_2} V^{i_1} V^{j_2} V^{k_2} V^{m_2} \\ + \Omega_{i_2 j_2 k_2 m_2} V^{i_2} V^{j_2} V^{k_2} V^{m_2}. \quad (124)$$

We now expand $\Omega(\bar{v})$ as a Taylor series with remainder:

$$\Omega(\bar{v}) = \Omega_0 + \bar{v}(D_v \Omega)_0 + \frac{1}{2}\bar{v}^2(D_v^2 \Omega)_0 + \frac{1}{6}\bar{v}^3(D_v^3 \Omega)_0 \\ + \frac{1}{6} \int_0^{\bar{v}} (\bar{v} - v)^3 D_v^4 \Omega dv, \quad (125)$$

where the subscript 0 indicates evaluation at $v = 0$. From the coincidence limits (69) and (121)–(123), we have

$$\Omega_0 = 0, \quad (D_v \Omega)_0 = 0, \quad (D_v^3 \Omega)_0 = 0, \quad (126)$$

and, with the aid of (17),

$$\bar{v}^2(D_v^2 \Omega)_0 = \overrightarrow{P_0 P_1^2} + \overrightarrow{P_0 P_2^2} - 2\overrightarrow{P_0 P_1} \cdot \overrightarrow{P_0 P_2}, \quad (127)$$

where these symbols are defined as follows:

$$\begin{aligned}\overrightarrow{P_0P_1^2} &= 2\Omega(P_0P_1), & \overrightarrow{P_0P_2^2} &= 2\Omega(P_0P_2), \\ \overrightarrow{P_0P_1} \cdot \overrightarrow{P_0P_2} &= \Omega_{i_0}(P_0P_1)\Omega^{i_0}(P_0P_2).\end{aligned}\tag{128}$$

This is a suggestive notation, but it must be used with some caution because Ω may be negative. If we were dealing with a space of positive-definite metric, then $\overrightarrow{P_0P_1^2}$ would be the square of the geodesic distance P_0P_1 , and $\overrightarrow{P_0P_1} \cdot \overrightarrow{P_0P_2}$ would be the product of the distances P_0P_1 , P_0P_2 and the cosine of the angle between the two directions.

In the above notation, (125) may be written

$$\overrightarrow{P_1P_2^2} = \overrightarrow{P_0P_1^2} + \overrightarrow{P_0P_2^2} - 2\overrightarrow{P_0P_1} \cdot \overrightarrow{P_0P_2} + \phi, \tag{129}$$

where

$$\phi = \frac{1}{3} \int_0^{\bar{v}} (\bar{v} - v)^3 D_v^4 \Omega dv. \tag{130}$$

This term ϕ is a 3-point invariant. It is the very essence of gravitational theory because it represents the deviation of space-time from flatness; if $\phi = 0$, then (129) becomes the Minkowskian form of the elementary trigonometrical formula

$$c^2 = a^2 + b^2 - 2ab \cos C. \tag{131}$$

In order to evaluate ϕ , we introduce an OT $\lambda_{(a)}^i$ which is chosen arbitrarily at P_0 and carried throughout space-time by parallel transport along all geodesics drawn out from P_0 . Taking components along this OT, (124) may be written

$$\begin{aligned}D_v^4 \Omega &= \Omega_{(a_1 b_1 c_1 d_1)} V^{(a_1)} V^{(b_1)} V^{(c_1)} V^{(d_1)} \\ &+ 4\Omega_{(a_1 b_1 c_1 d_2)} V^{(a_1)} V^{(b_1)} V^{(c_1)} V^{(d_2)} + \dots, \end{aligned} \tag{133}$$

in which the V 's are constants and $\Omega = \Omega(Q_1, Q_2)$. Are we entitled to substitute for the Ω -terms the values obtained in (117)? Not accurately, for to apply (117) we should use an OT with parallel transport along Q_1Q_2 in Fig. 4, and not along the geodesics drawn from P_0 . But it is easy to see that, to the order of approximation under consideration, we can substitute in (133) from (117), the error being absorbed into the O_2 residue. When we make this substitution, a number of terms disappear on account of the skew-symmetry of the

Riemann tensor hidden in the S -terms; in fact, no S -term survives if contracted, with respect to any three of its four leading indices, with the components of any single vector ($V^{(a_1)}$ or $V^{(a_2)}$): thus, for example,

$$S_{(abcq\bar{d})}V^{(a_1)}V^{(b_1)}V^{(c_1)} = 0. \quad (134)$$

To present the final result in simple form, we shall use a notation of which the following examples will be a sufficient explanation ¹:

$$\begin{aligned} [1122] &= S_{(abcd)}V^{(a_1)}V^{(b_1)}V^{(c_2)}V^{(d_2)} = [2211], \\ [112U2] &= S_{(abcq\bar{d})}V^{(a_1)}V^{(b_1)}V^{(c_2)}U^{(q)}V^{(d_2)}. \end{aligned} \quad (135)$$

Here is the result: *For a geodesic triangle (as in Fig. 4) in space-time of small curvature (O_1), we have the formula (129) with ϕ given by*

$$\begin{aligned} \phi &= \phi_0 + \phi_1 + \phi_2 + O_2, \\ \phi_0 &= 3k^3 \int_0^{\bar{v}} \int_{u_1}^{u_2} (\bar{v} - v)^3 \{ (u_2 - u)^2 + (u - u_1)^2 \} [1122] du dv, \\ \phi_1 &= 2k^3 \int_0^{\bar{v}} \int_{u_1}^{u_2} (\bar{v} - v)^3 \{ 2(u_2 - u)^3 [112U1] \\ &\quad + 3(u_2 - u)^2 (u - u_1) [112U2] \\ &\quad - 3(u_2 - u)(u - u_1)^2 [221U1] - 2(u - u_1)^3 [221U2] \} du dv, \\ \phi_2 &= \frac{1}{2}k^3 \int_0^{\bar{v}} \int_{u_1}^{u_2} (\bar{v} - v)^3 \{ (u_2 - u)^4 [11UU11] \\ &\quad + 4(u_2 - u)^3 (u - u_1) [11UU12] \\ &\quad + 3(u_2 - u)^2 (u - u_1)^2 ([11UU22] + [22UU11]) \\ &\quad + 4(u_2 - u)(u - u_1)^3 [22UU21] + (u - u_1)^4 [22UU22] \} du dv, \\ k^{-1} &= u_2 - u_1. \end{aligned} \quad (136)$$

The three parts into which ϕ is split involve respectively the Riemann tensor itself, its first-order covariant derivatives, and its second-order covariant derivatives.

We shall return to the above formula after introducing quasi-Cartesian coordinates in § 8.

§ 7. SOLUTION OF SMALL GEODESIC TRIANGLES

In the preceding section we dealt with a finite geodesic triangle in space-time of small curvature; now we consider a small geodesic

¹ These quantities are evaluated at a general point $Q(u, v)$ on Q_1Q_2 , with S and U calculated at Q and the V 's at Q_1 and Q_2 .

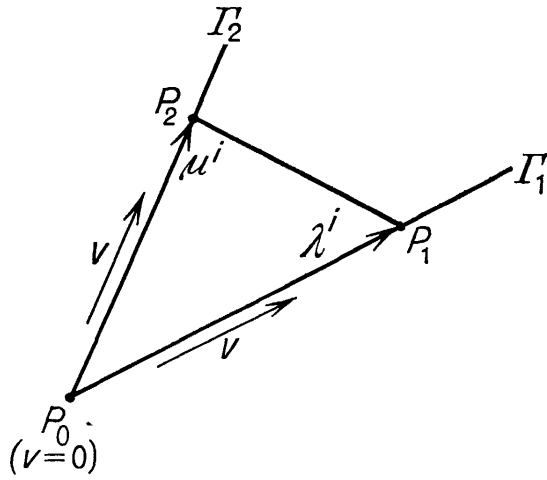


Fig. 5 – Solution of small geodesic triangle

triangle, and make no assumption about the curvature of space-time — it may be finite.

Let P_0 (Fig. 5) be any point in space-time, and let Γ_1 and Γ_2 be any two geodesics drawn out from P_0 . Let v be a special parameter on Γ_1 and on Γ_2 , with $v = 0$ at P_0 , and let P_1 and P_2 be points on Γ_1 and Γ_2 , respectively, corresponding to the same value of v . Then the world-function $\Omega(P_1P_2)$ is a function of v . We expand it in a series ¹

$$\begin{aligned} \Omega(P_1P_2) = \Omega(v) = \Omega_0 + v(D_v\Omega)_0 + \frac{1}{2}v^2(D_v^2\Omega)_0 \\ + \frac{1}{6}v^3(D_v^3\Omega)_0 + \frac{1}{24}v^4(D_v^4\Omega)_0 + O_5, \end{aligned} \quad (137)$$

where the subscript 0 indicates evaluation for $v = 0$, and O_5 indicates a term of the order of v^5 . The derivatives of Ω are precisely as written out in (121)–(124), V^{i_1} and V^{i_2} being the tangent vectors dx^i/dv on Γ_1 and Γ_2 , respectively. It is convenient to write

$$\lambda^i = v(V^{i_1})_0, \quad \mu^i = v(V^{i_2})_0. \quad (138)$$

When $v = 0$, P_1 and P_2 coincide at P_0 , and we can use the coincidence limits (69). Thus, with everything evaluated at P_0 , we find ²

$$\Omega(P_1P_2) = \Omega(v) = \frac{1}{2}(\lambda_i - \mu_i)(\lambda^i - \mu^i) + \frac{1}{4}S_{ijkl}\lambda^i\lambda^j\mu^k\mu^l + O_5. \quad (139)$$

Applying (2) and (17) and the notation (128), we have

$$\begin{aligned} \lambda_i\lambda^i &= 2\Omega(P_0P_1) = \overrightarrow{P_0P_1^2}, \\ \mu_i\mu^i &= 2\Omega(P_0P_2) = \overrightarrow{P_0P_2^2}, \\ \lambda_i\mu^i &= \Omega_{i_0}(P_0P_1)\Omega^{i_0}(P_0P_2) = \overrightarrow{P_0P_1} \cdot \overrightarrow{P_0P_2}, \end{aligned} \quad (140)$$

¹ If we wanted to be more precise mathematically, we would substitute for the residue O_5 an integral remainder as in (125). But we are here pursuing a different method of approximation, based on the smallness of v rather than on the smallness of the Riemann tensor.

² When we take the coincidence limit of (124), all terms except the middle one disappear on account of the skew-symmetry of the Riemann tensor.

and so (139) may be written

$$\overrightarrow{P_1P_2}^2 = \overrightarrow{P_0P_1}^2 + \overrightarrow{P_0P_2}^2 - 2\overrightarrow{P_0P_1} \cdot \overrightarrow{P_0P_2} + \psi + O_5, \quad (141)$$

where, by (48) and I-(99),

$$\begin{aligned} \psi &= \frac{1}{2}S_{ijkm}\lambda^i\lambda^j\mu^k\mu^m \\ &= -\frac{1}{3}R_{ijkm}\lambda^i\mu^j\lambda^k\mu^m \\ &= -\frac{1}{3}K(\lambda^i\lambda_i\mu^j\mu_j - (\lambda^i\mu_i)^2) \\ &= -\frac{1}{3}K(\overrightarrow{P_0P_1}^2\overrightarrow{P_0P_2}^2 - (\overrightarrow{P_0P_1} \cdot \overrightarrow{P_0P_2})^2), \end{aligned} \quad (142)$$

K being the Riemannian curvature of space-time associated with any 2-element of the small triangle $P_0P_1P_2$.

Since ψ is O_4 , we can afford to treat the triangle as if space-time were flat, the error so caused being absorbed into the O_5 residue. Since three null lines cannot form a triangle in flat space-time, at least one side of $P_0P_1P_2$ is not null. Let ε denote the indicator of that side and ε' the indicator of the perpendicular dropped on that side from the opposite vertex. Then it is easy to show that, for any geodesic triangle in flat space-time,

$$\overrightarrow{P_0P_1}^2\overrightarrow{P_0P_2}^2 - (\overrightarrow{P_0P_1} \cdot \overrightarrow{P_0P_2})^2 = 4\varepsilon\varepsilon'\Delta^2, \quad (143)$$

where Δ is the 2-area of the triangle. Thus *for a small geodesic triangle we have the formula*

$$\overrightarrow{P_1P_2}^2 = \overrightarrow{P_0P_1}^2 + \overrightarrow{P_0P_2}^2 - 2\overrightarrow{P_0P_1} \cdot \overrightarrow{P_0P_2} - \frac{4}{3}K\varepsilon\varepsilon'\Delta^2 + O_5 \quad (144)$$

or, equivalently,

$$\begin{aligned} \Omega(P_1P_2) &= \Omega(P_0P_1) + \Omega(P_0P_2) - \Omega_{i_0}(P_0P_1)\Omega^{i_0}(P_0P_2) \\ &\quad - \frac{2}{3}K\varepsilon\varepsilon'\Delta^2 + O_5. \end{aligned} \quad (145)$$

It is interesting to compare (129) (finite triangle, small curvature) with (141) (small triangle, finite curvature). The leading terms agree, and we have ϕ in (129) where we have ψ in (141). To reconcile the two formulae, let the triangle in (129) become small. That means letting $u_2 \rightarrow u_1$ in (136): then ϕ_1 and ϕ_2 disappear, while we find immediately that ϕ_0 reduces to ψ as written in (142) in terms of the S -tensor.

§ 8. QUASI-CARTESIAN COORDINATES

In flat space-time there exist coordinates such that, throughout space-time, the metric tensor is

$$g_{ij} = \eta_{ij} = \text{diag}(1, 1, 1, -1), \quad (146)$$

and the world-function reads

$$\Omega(x', x) = \frac{1}{2}\eta_{ij}(x^{i'} - x^i)(x^{j'} - x^j). \quad (147)$$

If space-time is nearly flat (small Riemann tensor), it is natural to assume the existence of coordinates such that

$$g_{ij} = \eta_{ij} + \gamma_{ij}, \quad (148)$$

where the γ 's and their derivatives are small. This procedure has been

widely used in relativity, but some obscurity exists regarding the class of coordinates for which the approximation is valid ¹, and we shall approach the matter in a different and more definite way.

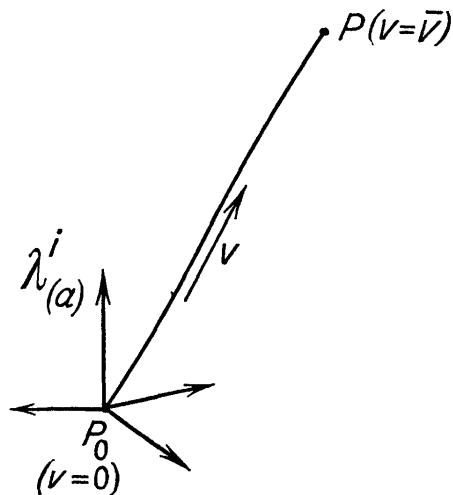


Fig. 6 – Origin P_0 and vector base $\lambda^i_{(a)}$ for quasi-Cartesian coordinates

Let P_0 (Fig. 6) be any point in space-time and $\lambda^i_{(a)}$ an orthonormal tetrad (OT) at P_0 . We call P_0 the *origin* and the tetrad the *vector base*. Let P be any other point such that there exists a unique geodesic P_0P , and let v be a special parameter on this geodesic, taking the values $v = 0$ at P_0 and $v = \bar{v}$ at P . Then there exists a vector $\bar{v}(dx^i/dv)$

tangent to the geodesic at P_0 , and we denote its contravariant components [cf. I-(54)] on the vector base as follows:

$$X^{(a)} = \bar{v} \left(\frac{dx^i}{dv} \right)_{P_0} \lambda^i_{(a)}. \quad (149)$$

In view of (17), the components may also be expressed in terms of the derivatives of the world-function $\Omega(P_0P)$, in both covariant and

¹ In Euclidean geometry, the curvature of a sphere of large radius is small. But there exists no coordinate system covering the whole sphere for which a formula like (148) holds with small γ 's; we would have to use a great number of overlapping coordinate systems.

contravariant forms as follows:

$$\begin{aligned} X_{(a)} &= -\Omega_{i_0}(P_0P)\lambda_{(a)}^i, \\ X^{(a)} &= -\Omega^{i_0}(P_0P)\lambda_i^{(a)}. \end{aligned} \quad (150)$$

These are 2-point invariants with respect to general coordinate transformations. We call them the *quasi-Cartesian coordinates*¹ (briefly QC) of P relative to the origin P_0 and the vector base $\lambda_{(a)}^i$.

It is easy to see that

$$X^{(a)}X_{(a)} = 2\Omega(P_0P) = \varepsilon L^2, \quad (151)$$

where ε is the indicator of P_0P and L its measure. For a pair of points P_1, P_2 , we find²

$$X^{(a_1)}X_{(a_2)} = \Omega_{i_0}(P_0P_1)\Omega^{i_0}(P_0P_2). \quad (152)$$

In ordinary geometry, a change of the origin of rectangular Cartesian coordinates is trivial, but the effect of rotating the axes is, by contrast, rather sophisticated; it involves orthogonal matrices, Eulerian angles, etc. In curved space-time, on the other hand, a change of vector base is comparatively trivial (a mere Lorentz transformation), but a change of origin is a much more delicate affair. Postponing the change of origin to § 9, we may describe the effect of changing the vector base from $\lambda_{(a)}^i$ to $\mu_{(a)}^i$ briefly as follows. Let X and Y refer to the corresponding QC of some point P . Then, by 1-(52) we obtain

$$Y_{(b)} = L_{(b)}^{(a)}X_{(a)}, \quad Y^{(b)} = \eta^{(bc)}L_{(c)}^{(a)}\eta_{(ad)}X^{(d)}, \quad (153)$$

where

$$L_{(b)}^{(a)} = \lambda_i^{(a)}\mu_{(b)}^i, \quad (154)$$

the Lorentz matrix as in 1-(51).

In Newtonian physics, the power of the usual vector notation lies in the suppression of the axes on which the components of the vector are to be taken. We think of \overrightarrow{PQ} as a geometric object, a thing-in-itself. If we exercise some caution, we may likewise think of the dis-

¹ They are also known as *normal coordinates*, but their importance warrants a more striking name.

² This compact notation should not cause confusion. $X^{(a_1)}$ and $X_{(a_2)}$ are given by (150) on substituting P_1 and P_2 respectively for P , and there is summation with respect to the Lorentz index a . If we chose to write $X^{(a)}$ for the QC of P_1 and $Y_{(a)}$ for those of P_2 , then the left hand side of (152) would read $X^{(a)}Y_{(a)}$.

placement from P_0 to P_1 in space-time as a thing-in-itself, without bothering about the vector base which is used in (150). Thus we boldly write $\overrightarrow{P_0P}$ as a symbol to stand for $X^{(a)}$ or $X_{(a)}$, and this fits in admirably with the notation already introduced in (128), for in that notation we have

$$\overrightarrow{P_0P^2} = X^{(a)}X_{(a)}, \quad \overrightarrow{P_0P_1} \cdot \overrightarrow{P_0P_2} = X^{(a_1)}X_{(a_2)}, \quad (155)$$

as we see from (151) and (152). These quantities are obviously independent of the vector base, since the vector base does not appear in (151) and (152).

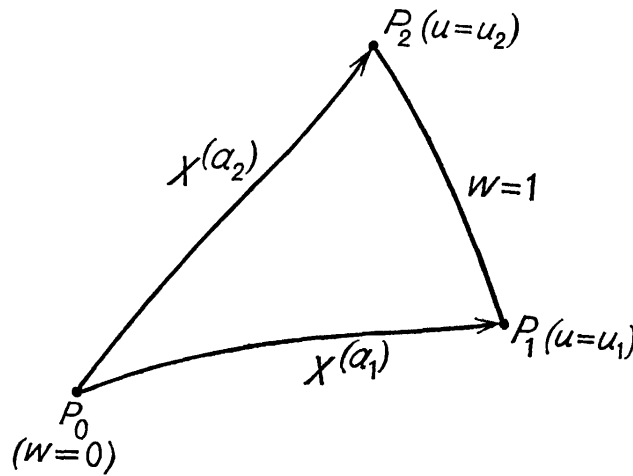


Fig. 7 – Finite geodesic triangle and quasi-Cartesian coordinates

Let us now return to the finite geodesic triangle discussed in § 6, and examine the solution with the aid of QC. Taking P_0 (Fig. 7) as the origin of QC, the formula (129) may be written

$$\overrightarrow{P_1P_2^2} = \xi^{(a)}\xi_{(a)} + \phi, \quad (156)$$

where ¹

$$\xi^{(a)} = X^{(a_2)} - X^{(a_1)}. \quad (157)$$

For ϕ we have the expression (136), with the notation (135). We are now to put

$$\bar{v}V^{(a_1)} = X^{(a_1)}, \quad \bar{v}V^{(a_2)} = X^{(a_2)}. \quad (158)$$

Further, it is easy to see that in the approximation for ϕ we may

¹ We must *not* write $\overrightarrow{P_1P_2}$ for $\xi^{(a)}$; that would be a fatal mistake.

think of the triangle as lying in flat space-time, and put

$$U^{(q)} = (v/\bar{v})\xi^{(q)}. \quad (159)$$

We modify the notation (135), writing ¹

$$\begin{aligned} \{1122\} &= S_{(abcd)}X^{(a_1)}X^{(b_1)}X^{(c_2)}X^{(d_2)} = \{2211\}, \\ \{11222\} &= S_{(abcde)}X^{(a_1)}X^{(b_1)}X^{(c_2)}X^{(d_2)}X^{(e_2)} = \{22112\}, \end{aligned} \quad (160)$$

and so on. Then, if we write $v/\bar{v} = w$, we find that the solution of a finite geodesic triangle in space-time of small curvature is

$$\begin{aligned} \overrightarrow{P_1 P_2} &= \xi^{(a)}\xi_{(a)} + \phi, \\ \phi &= \phi_0 + \phi_1 + \phi_2 + O_2, \\ \phi_0 &= 3k^3 \int_0^1 (1-w)^3 dw \int_{u_1}^{u_2} [(u_2 - u)^2 + (u - u_1)^2] \{1122\} du, \\ \phi_1 &= 2k^3 \int_0^1 w(1-w)^3 dw \int_{u_1}^{u_2} [2(u_2 - u)^3 \{11221\} \\ &\quad + 3(u_2 - u)^2(u - u_1) \{11222\} + 3(u_2 - u)(u - u_1)^2 \{22111\} \\ &\quad + 2(u - u_1)^3 \{22112\}] du, \\ \phi_2 &= \frac{1}{2}k^3 \int_0^1 w^2(1-w)^3 dw \int_{u_1}^{u_2} [(u_2 - u)^4 \{112211\} \\ &\quad + 4(u_2 - u)^3(u - u_1) \{112212\} \\ &\quad + 3(u_2 - u)^2(u - u_1)^2 (\{112222\} + \{221111\}) \\ &\quad + 4(u_2 - u)(u - u_1)^3 \{221121\} + (u - u_1)^4 \{221122\}] du, \\ k^{-1} &= u_2 - u_1. \end{aligned} \quad (161)$$

There are other equivalent forms. On account of the symmetry of the S-tensor, we can change a leading 1122 into 2211. Further, since we are at the limit of explicit approximation, we can treat covariant differentiation in ϕ as if it were partial differentiation, and so we can interchange the last two numbers in a 6-index symbol. For example,

$$\begin{aligned} \{1122\} &= \{2211\}, & \{11221\} &= \{22111\}, \\ \{112212\} &= \{112221\} + O_2. \end{aligned} \quad (162)$$

¹ Any one of these symbols vanishes if it has three or four like numbers in the first four places. This is due to the skew-symmetry of the Riemann tensor.

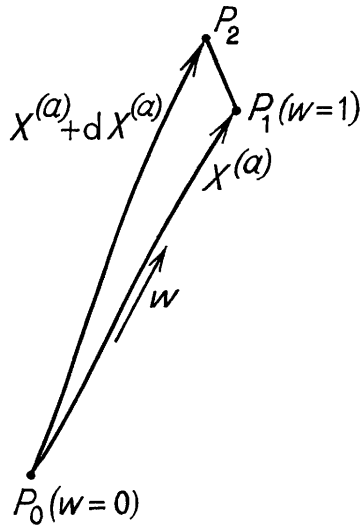


Fig. 8 – Metric in terms
of Riemann tensor

We shall now find an approximate expression for the metric tensor for QC in a space-time of small curvature. The result could be obtained from (161) by letting u_2 tend to u_1 , so that the triangle $P_0P_1P_2$ becomes very thin, in Fig. 8.

But it is better to go back to the deviation equation and (91), changing the notation to suit Fig. 4 (interchange u and v). If \bar{u} denotes the infinitesimal increment in u in passing from P_0P_1 to P_1P_2 , the QC of P_1, P_2 may be written $X_{(a)}, X_{(a)} + dX_{(a)}$ respectively, where

$$X_{(a)} = \bar{v}V_{(a_0)}, \quad dX_{(a)} = \bar{v}\bar{u} \frac{\partial}{\partial u} V_{(a_0)}. \quad (163)$$

When applied to the geodesics P_0P_1, P_0P_2 of Fig. 4, brought close together as in Fig. 8, (91) gives

$$\frac{\partial}{\partial u} V_{(a_0)} = \bar{v}^{-1}U_{(a_2)} + \bar{v}^{-2}V^{(b)}U^{(c_2)}V^{(d)} \int_0^{\bar{v}} v(\bar{v} - v)R_{(abcd)}dv + O_2. \quad (164)$$

When multiplied by $\bar{u}\bar{v}$, this may be written

$$\bar{u}U_{(a_2)} = dX_{(a)} - X^{(b)}dX^{(c)}X^{(d)} \int_0^1 w(1 - w)R_{(abcd)}dw + O_2, \quad (165)$$

changing to the parameter $w = v/\bar{v}$. Hence we have for *the metric form of space-time for the quasi-Cartesian coordinates*

$$\begin{aligned} \Phi &= \bar{u}^2 U_{(a_2)} U^{(a_2)} = g_{(ab)} dX^{(a)} dX^{(b)}, \\ g_{(ab)} &= \eta_{(ab)} + \gamma_{(ab)}, \\ \gamma_{(ab)} &= -2X^{(c)}X^{(d)} \int_0^1 w(1 - w)R_{(acbd)}dw + O_2. \end{aligned} \quad (166)$$

In the particular case of a space-time with small constant curvature K , we have

$$R_{(acbd)} = K(\eta_{(ab)}\eta_{(cd)} - \eta_{(ad)}\eta_{(cb)}), \quad (167)$$

and hence

$$\gamma_{(ab)} = \frac{1}{3}K(X_{(a)}X_{(b)} - \eta_{(ab)}X^{(c)}X_{(c)}) + O_2. \quad (167a)$$

At the origin P_0 of the QC we have, by (166),

$$g_{(ab)} = \eta_{(ab)}, \quad \frac{\partial g_{(ab)}}{\partial X^{(c)}} = 0, \quad (168)$$

$$\frac{\partial^2 g_{(ab)}}{\partial X^{(c)} \partial X^{(d)}} = -2(R_{(acb)d} + R_{(adb)c}) \int_0^1 w(1-w)dw = S_{(abcd)}.$$

§ 9. CHANGING THE ORIGIN OF QUASI-CARTESIAN COORDINATES

The construction of quasi-Cartesian coordinates (QC), as in § 8, requires that there should be a unique geodesic joining the origin P_0 to the current point P . If there are two or more joining geodesics, the construction fails, and indeed the whole approximate treatment of geodesic deviation, as we have used it, breaks down. Now it may be possible to cover the whole of space-time with a QC system (it

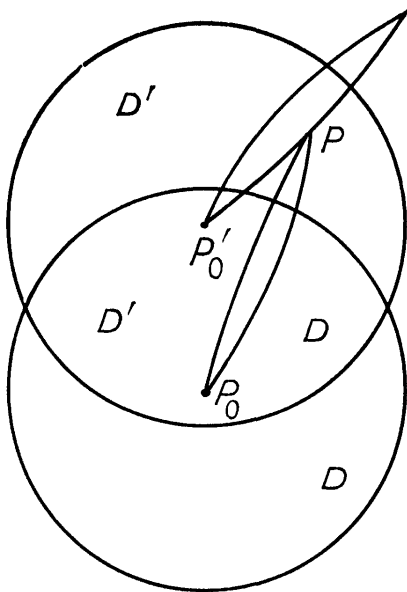


Fig. 9 – Continuation of quasi-Cartesian coordinates

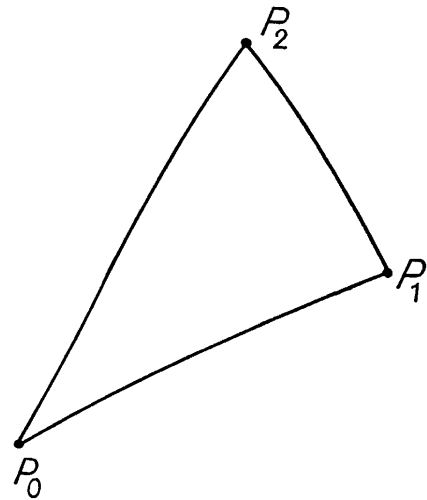


Fig. 10 – Changing the origin of quasi-Cartesian coordinates from P_0 to P_1

certainly is possible if space-time is flat), but to cope with physical situations which actually arise it is desirable to suppose that geodesics drawn from the origin P_0 may intersect ¹. There will however be some domain D containing P_0 without intersections, and in D the

¹ This means that we here face up to the existence of conjugate points: cf. II-§ 3.

QC may be set up (Fig. 9). To extend the QC, we then choose another point in D , say P'_0 , and use it as origin of new QC with domain D' . In this way we can carry on the QC indefinitely.

Accordingly we seek formulae for the transformation of QC corresponding to change of origin. As remarked earlier, this is a trivial transformation in flat space-time, but a much more complicated matter when space-time is curved.

Let P_0 and P_1 be two origins of QC (Fig. 10) and P_2 a current point lying in the domains of validity of both QC. To avoid any possible ambiguity, we write very explicitly

$$\begin{aligned} \text{QC of } P_0 \text{ relative to } P_1 &= X_{(a)}(\overrightarrow{P_1 P_0}) = -\Omega_{i_1}(P_1 P_0)\lambda_{(a)}^{i_1}, \\ \text{QC of } P_2 \text{ relative to } P_0 &= X_{(a)}(\overrightarrow{P_0 P_2}) = -\Omega_{i_0}(P_0 P_2)\lambda_{(a)}^{i_0}, \\ \text{QC of } P_2 \text{ relative to } P_1 &= X_{(a)}(\overrightarrow{P_1 P_2}) = -\Omega_{i_1}(P_1 P_2)\lambda_{(a)}^{i_1}. \end{aligned} \quad (169)$$

Since the effect of changing the vector base is trivial (a mere Lorentz transformation), we shall suppose for simplicity that $\lambda_{(a)}^{i_0}$ is parallel (along $P_0 P_1$) to $\lambda_{(a)}^{i_1}$, so that, in terms of the parallel propagator,

$$\lambda_{(a)i_0} = g_{i_0 j_1} \lambda_{(a)}^{j_1}. \quad (170)$$

We have then from (169)

$$X_{(a)}(\overrightarrow{P_1 P_2}) - X_{(a)}(\overrightarrow{P_1 P_0}) - X_{(a)}(\overrightarrow{P_0 P_2}) = \theta_{(a)}, \quad (171)$$

where

$$\theta_{(a)} = \lambda_{(a)}^{j_1} [-\Omega_{j_1}(P_1 P_2) + \Omega^{i_0}(P_0 P_2)g_{i_0 j_1} + \Omega_{j_1}(P_1 P_0)]. \quad (172)$$

Define the 2-point tensor $h_{i_0 j_1}$ by

$$h_{i_0 j_1} = g_{i_0 j_1} + \Omega_{i_0 j_1}(P_0 P_1), \quad (173)$$

and define the 3-point invariant χ by

$$\chi = \Omega(P_1 P_2) - \Omega(P_0 P_1) - \Omega(P_0 P_2) + \Omega_{i_0}(P_0 P_1)\Omega^{i_0}(P_0 P_2), \quad (174)$$

so that, on differentiation with respect to P_1 ,

$$\chi_{j_1} = \Omega_{j_1}(P_1 P_2) - \Omega_{j_1}(P_0 P_1) + \Omega_{i_0 j_1}(P_0 P_1)\Omega^{i_0}(P_0 P_2). \quad (175)$$

Then (172) may be written

$$\theta_{(a)} = \lambda_{(a)}^{j_1} [-\chi_{j_1} + h_{i_0 j_1} \Omega^{i_0}(P_0 P_2)]. \quad (176)$$

This value is to be inserted in (171), which may be written as follows to show the *law of transformation of quasi-Cartesian coordinates when the origin is changed from P_0 to P_1* :

$$\overrightarrow{X_{(a)}(P_1P_2)} = \overrightarrow{X_{(a)}(P_1P_0)} + \overrightarrow{X_{(a)}(P_0P_2)} + \theta_{(a)}. \quad (177)$$

The significance of this formula lies in the fact that we have actually computed $\theta_{(a)}$ approximately for space-time of small curvature. Thus (95) gives, with appropriate changes in notation,

$$h_{i_0j_1} = \frac{3}{2} \overrightarrow{X^{(c)}(P_0P_1)} \overrightarrow{X^{(d)}(P_0P_1)} \int_0^1 v(1-v) S_{(ijcd)} dv + O_2, \quad (178)$$

where the integration is taken along P_0P_1 with respect to a special parameter v which takes the values zero and unity at P_0 and P_1 respectively. As for χ , on comparing (174) with (161), we see that,

$$\chi = \frac{1}{2}\phi + O_2, \quad (179)$$

where ϕ is given in (161) by a complicated formula, to be interpreted with the aid of Fig. 7.

§ 10. FERMI COORDINATES AND OPTICAL COORDINATES

Quasi-Cartesian coordinates, as discussed in the preceding sections, involve a choice of origin and vector base, and it is not easy in physical problems to pick out a suitable origin. We have seen, moreover, that although a change of vector base is a trivial matter, a change of origin is complicated. From a physical standpoint it is usually better to use coordinates based on a timelike curve instead of on a single point ¹.

Let C be a timelike curve and P_0 some point chosen on it (Fig. 11). Let $\lambda_{(a)}^i$ be an orthonormal tetrad carried along C by Fermi-Walker transport, with $\lambda_{(4)}^i$ tangent to C . Let A^i , B^i and b be respectively the unit tangent to C , its first unit normal, and its first curvature, so that,

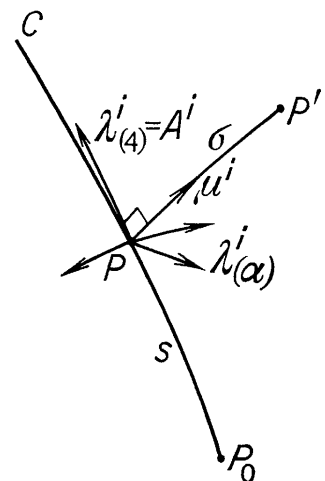


Fig. 11 – Fermi coordinates relative to a timelike base line C

¹ In physical language, the timelike curve is the history, or world-line, of an observer. As we shall see in later chapters, the mathematics of the present section lies close to physical experience.

as in I-(84),

$$\frac{\delta}{\delta s} \lambda_{(\alpha)}^i = b A^i \lambda_{(\alpha)}^j B_j, \quad \lambda_{(4)}^i = A^i, \quad (180)$$

Greek suffixes taking the values 1, 2, 3, as usual; here s is the measure of C from P_0 to the current point.

Let P' be any point in space-time such that a unique geodesic may be drawn from P' to cut C orthogonally (say at P). Let s and σ be the measures of P_0P and PP' , respectively, and let μ^i be the unit tangent vector to PP' at P . (PP' is of course spacelike, since it is orthogonal to the timelike C .) We then define the *contravariant Fermi coordinates*¹ (briefly FC) of P' relative to the base line C by

$$X^{(\alpha)} = \sigma \mu^i \lambda_i^{(\alpha)}, \quad X^{(4)} = s, \quad (181)$$

and the covariant FC by

$$X_{(\alpha)} = \eta_{(\alpha b)} X^{(b)} = X^{(\alpha)}, \quad X_{(4)} = \eta_{(4b)} X^{(b)} = -X^{(4)}. \quad (182)$$

We note that, for the first three coordinates, the contravariant and covariant forms are the same; for the fourth coordinate, there is a change of sign.

The FC are *2-point-curve invariants* in the sense that they are invariants determined (in a given space-time) by the points P' and P_0 and the curve C , which may be a geodesic, but in general is not.

Under change of base line, the FC transform in a very complicated way which we shall not attempt to discuss here. But if we retain the base line and merely shift P_0 along it, then the change is trivial (add a constant to $X^{(4)}$). Further, the change is very simple if we alter the triad $\lambda_{(\alpha)}^i$; for it can be subjected only to a rotation with *constant* coefficients, say

$$\bar{\lambda}_{(\alpha)}^i = H_{(\alpha)}^{(\beta)} \lambda_{(\beta)}^i, \quad H_{(\alpha)}^{(\beta)} H_{(\gamma)}^{(\beta)} = \delta_{\alpha\gamma}, \quad (183)$$

and the corresponding transformation is

$$\bar{X}_{(\alpha)} = H_{(\alpha)}^{(\beta)} X_{(\beta)}, \quad (184)$$

with $X^{(4)}$ unchanged.

On comparing (18) and (181), we see that the first three covariant

¹ The term 'Fermi coordinates' is sometimes used in a different sense, viz. to describe coordinates such that the Christoffel symbols vanish on some curve or other subspace: cf. FERMI [1922], O'RAIFEARTAIGH [1958b].

FC can be expressed neatly in terms of the world-function $\Omega(PP')$ as follows:

$$X_{(\alpha)} = -\Omega_i(PP')\lambda_{(\alpha)}^i. \quad (185)$$

We have

$$X_{(\alpha)}X^{(\alpha)} = 2\Omega(PP') = \sigma^2. \quad (186)$$

If we indicate by a bar the value of a quantity when the contra-variant FC are used as a coordinate system ($X^{(a)} = \bar{x}^a$), we have

$$\bar{\lambda}_{(\alpha)}^i = \delta_{\alpha}^i, \quad \bar{A}^{\alpha} = 0, \quad \bar{A}^4 = 1. \quad (187)$$

It is easy to prove, using the geodesic character of PP' , (180) and I-(55a), that the following equations hold on C (in the notation of I-(45)):

$$\bar{g}_{ij} = \eta_{ij}, \quad \bar{\Gamma}_{\alpha\beta}^i = 0, \quad \bar{\Gamma}_{\beta 4}^{\alpha} = 0, \quad \bar{\Gamma}_{\alpha 4}^4 = \bar{\Gamma}_{44}^{\alpha}, \quad \bar{\Gamma}_{44}^4 = 0. \quad (188)$$

If in particular C happens to be a geodesic, then on C (comma means partial derivative)

$$\bar{g}_{ij} = \eta_{ij}, \quad \bar{g}_{ij,k} = 0, \quad \bar{\Gamma}_{jk}^i = 0. \quad (188a)$$

From a geometrical standpoint, the Fermi coordinates considered above are the simplest coordinates we can define in terms of a timelike base line C . But from a physical standpoint, a spacelike geodesic PP' is somewhat artificial, and it is advantageous to use a different plan, replacing the spacelike geodesic by a null geodesic.

Fig. 12 shows a timelike curve C with a point P_0 on it from which s is measured. Let $\lambda_{(\alpha)}^i$ be an OT with Fermi-Walker transport along C and with $\lambda_{(4)}^i = A^i$, the unit tangent vector to C , precisely as in Fig. 11. Let P' be any point in space-time. From P' draw the future sheet of the null cone cutting C at P , say.

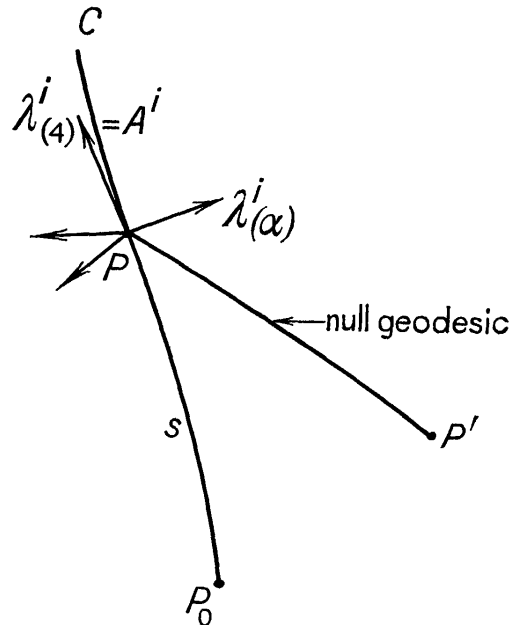


Fig. 12 - Optical coordinates relative to a timelike base line C

We define the contravariant and covariant *optical coordinates*¹ (briefly OC) of P' relative to the base line C by

$$X^{(\alpha)} = X_{(\alpha)} = -\Omega_i(PP')\lambda_{(\alpha)}^i, \quad X^{(4)} = -X_{(4)} = s, \quad (189)$$

where s is the measure of P_0P . Note that, if we use (185) as the definition of the first three FC, the equations defining FC and OC are formally identical. The difference consists in the fact that in FC the geodesic PP' is *orthogonal* to the base line C , whereas in OC this geodesic is *null*. These facts may be expressed as follows:

$$\Omega_i(PP')A^i = 0 \text{ for FC}, \quad (190a)$$

$$\Omega(PP') = 0 \text{ for OC}. \quad (190b)$$

Thus the world-function plays an important role in unifying and simplifying the treatment of these coordinates².

To complete the comparison of FC and OC, we note that (181) does not hold for OC because the measure of PP' is zero. But if we introduce a special parameter u on the null geodesic PP' , running from $u = 0$ at P to $u = \sigma$ at P' (σ being any number), then (189) may be written

$$X^{(\alpha)} = \sigma \lambda_{(\alpha)}^i \left(\frac{dx^i}{du} \right)_P. \quad (191)$$

The equation (186) is false for OC. To find the corresponding equation, we observe that

$$\lambda_{(\alpha)}^i \lambda_{(\alpha)}^j = \delta_i^j - \lambda_{(4)}^i \lambda_{(4)}^j = \delta_i^j + A_i A^j, \quad (192)$$

and hence

$$X_{(\alpha)} X^{(\alpha)} = \Omega_j \lambda_{(\alpha)}^j \Omega^i \lambda_{(\alpha)}^i = \Omega_j \Omega^i (\delta_i^j + A_i A^j) = (\Omega_i A^i)^2, \quad (193)$$

since [cf. (20)] $\Omega_i \Omega^i = 2\Omega = 0$.

In flat space-time, for both FC and OC, $X_{(\alpha)}$ are very simple; they are the spatial Cartesian coordinates of P' for moving axes for which the origin is at P and the x^4 -axis is tangent to C .

¹ These optical coordinates differ from those defined by TEMPLE [1938]; his definition involved parallel transport of null vectors and not Fermi transport of a reference triad.

² Although we use the word *coordinates* for FC and OC, we should remember that they are really invariants, and that a general coordinate system is lurking always in the background; by its use we avoid ugly unsymmetric expressions such as (188).

§ 11. METRICS FOR FERMI COORDINATES AND OPTICAL COORDINATES

In order to investigate the metrics for Fermi coordinates (FC) and optical coordinates (OC), we start with the general situation shown in Fig. 13. We have a timelike curve C_1 and a second curve C_2 (not necessarily timelike), and these two curves are joined by a single infinity of geodesics, each of which has a special parameter u running from $u = u_1$ on C_1 to $u = u_2$ on C_2 . We label these geodesics with a parameter v which is equal to the measure s of C_1 from some chosen point P_0 on it. We have then a 2-space $x^i = x^i(u, v)$, and as usual we write $U^i = \partial x^i / \partial u$, $V^i = \partial x^i / \partial v$.

For the partial derivative of the world-function $\Omega(P_1 P_2)$ with respect to P_1 we have

$$\begin{aligned} \Omega_{i_1} &= -k^{-1} U_{i_1}, \\ k^{-1} &= u_2 - u_1, \end{aligned} \quad (194)$$

and differentiation with respect to v gives

$$\Omega_{i_1 j_1} V^{j_1} + \Omega_{i_1 j_2} V^{j_2} = -W_{i_1}, \quad W_{i_1} = k^{-1} \delta U_{i_1} / \delta v. \quad (195)$$

Now for a space-time of small curvature (O_1), we have

$$\Omega_{i_1 j_1} = g_{i_1 j_1} + h_{i_1 j_1}, \quad \Omega_{i_1 j_2} = -g_{i_1 j_2} + h_{i_1 j_2}, \quad (196)$$

where $g_{i_1 j_2}$ is the parallel propagator and the h -terms are small integrals given in (95). Hence

$$g_{i_1 j_2} V^{j_2} = W_{i_1} + V_{i_1} + h_{i_1 j_1} V^{j_1} + h_{i_1 j_2} V^{j_2}. \quad (197)$$

The left hand side is the result of taking V^{j_2} from P_2 to P_1 by parallel transport, and this does not change the magnitude of the vector. Therefore if we write for brevity

$$W^i + V^i = Z^i, \quad (198)$$

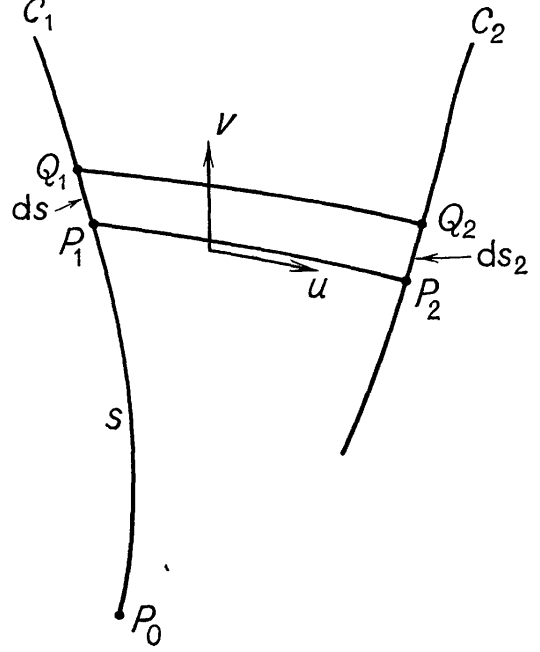


Fig. 13 – Determination of metrics for Fermi coordinates and optical coordinates

we have (since the h -terms are O_1)

$$V_{i_2}V^{i_2} = Z_{i_1}Z^{i_1} + 2Z^{i_1}(h_{i_1j_1}V^{j_1} + h_{i_1j_2}V^{j_2}) + O_2. \quad (199)$$

By (197)

$$V^{j_2} = g^{j_2k_1}Z_{k_1} + O_1, \quad (200)$$

and so (199) may be written

$$V_{i_2}V^{i_2} = Z_{i_1}Z^{i_1} + 2Z^{i_1}h_{i_1j_1}V^{j_1} + 2Z^{i_1}h_{i_1j_2}g^{j_2k_1}Z_{k_1} + O_2. \quad (201)$$

All the vectors on the right hand side are now evaluated at P_1 .

If the geodesics P_1P_2 and Q_1Q_2 (Fig. 13) correspond to values v and $v + dv$ (i.e. s and $s + ds$), then

$$V_{i_2}V^{i_2}ds^2 = \overrightarrow{P_2Q_2^2} = 2\Omega(P_2Q_2) = \varepsilon ds_2^2, \quad (202)$$

where ds_2 is the measure of P_2Q_2 and ε is its indicator. Thus we have only to multiply (201) by ds^2 in order to obtain the metric form at P_2 , assuming that Z^i and V^i are known on C_1 . Actually we know that on C_1 , $V^i = A^i$, the unit tangent vector to C_1 ; the vector Z^i on the other hand depends on the choice of the curve C_2 .

In both FC and OC we have an OT $\lambda_{(a)}^i$ which undergoes F-W transport along the base line, which is C_1 in Fig. 13. We have also $\lambda_{(4)}^i = A^i$. Let us define an OT over the 2-space of geodesics in Fig. 13 by parallel transport along the geodesics, and let us express (201) in terms of invariant components on this OT. We can, without confusion, drop the secondary numerical suffix on the right hand side in the case of the vectors (they are evaluated at P_1), but we must retain them for the h -terms because they involve both P_1 and P_2 . We have

$$V^{(\alpha)} = 0, \quad V^{(4)} = 1, \quad V_{(4)} = -1, \quad (203)$$

and so we get

$$2\Omega(P_2Q_2) = ds^2[Z_{(a)}Z^{(a)} + 2Z^{(a)}h_{(a_14_1)} + 2Z^{(a)}h_{(a_1b_2)}Z^{(b)}] + O_2. \quad (204)$$

Taking C_1 for base line, and understanding all vectors to be evaluated on it (so that the secondary suffix 1 is not required), we have for both FC and OC

$$X_{(\alpha)} = k^{-1}U_i\lambda_{(\alpha)}^i = k^{-1}U_{(\alpha)}, \quad X^{(4)} = s, \quad (205)$$

and hence, with $D = d/ds = d/dv$,

$$DX_{(\alpha)} = W_{(\alpha)} + k^{-1}U_iA^ibB_{(\alpha)}. \quad (206)$$

Let us now pursue FC alone, leaving OC till later. By the ortho-

gonality shown in Fig. 11, we have for FC

$$U_i A^i = 0, \quad (207)$$

and differentiation gives

$$kW_i A^i + bU_i B^i = 0, \quad W_{(4)} = -bX_{(\alpha)}B^{(\alpha)}. \quad (208)$$

Hence, by (203), (206) and (208),

$$Z^{(\alpha)} = DX^{(\alpha)}, \quad Z^{(4)} = 1 + \zeta = -Z_{(4)}, \quad (209)$$

where

$$\zeta = bX_{(\alpha)}B^{(\alpha)}. \quad (210)$$

(We recall that b is the curvature of the base line and B^i its unit first normal vector.)

Substitution from (203) and (209) in (204) gives the following expression for *the metric form for Fermi coordinates in space-time of small curvature*:

$$\begin{aligned} 2\Omega(P_2Q_2) &= \Phi = g_{(rs)}dX^{(r)}dX^{(s)}, \\ g_{(\alpha\beta)} &= \delta_{\alpha\beta} + 2h_{(\alpha_1\beta_2)} + O_2, \\ g_{(\alpha 4)} &= h_{(\alpha_1 4_1)} + 2h_{(\alpha_1 4_2)}(1 + \zeta) + O_2, \\ g_{(44)} &= -(1 + \zeta)^2 + 2(1 + \zeta)h_{(4_1 4_1)} + 2(1 + \zeta)^2h_{(4_1 4_2)} + O_2. \end{aligned} \quad (211)$$

As regards the small h -terms, they occur in (97) as integrals of the symmetrized Riemann S -tensor. To express them in terms of the FC, we have to alter the notation, taking the special parameter u in (97) to be the measure on the geodesic, so that $u_1 = 0$ and $u_2 = \sigma$. Further, let us write for brevity

$$\frac{3}{2}\sigma^{-3}X^{(\mu)}X^{(\nu)} = Y^{(\mu\nu)}. \quad (212)$$

Then we have

$$\begin{aligned} h_{(\alpha_1\beta_2)} &= Y^{(\mu\nu)} \int_0^\sigma (\sigma - u)u S_{(\alpha\beta\mu\nu)} du + O_2, \\ h_{(\alpha_1 4_1)} &= Y^{(\mu\nu)} \int_0^\sigma (\sigma - u)^2 S_{(\alpha 4\mu\nu)} du + O_2, \\ h_{(\alpha_1 4_2)} &= h_{(\alpha_2 4_1)} = Y^{(\mu\nu)} \int_0^\sigma (\sigma - u)u S_{(\alpha 4\mu\nu)} du + O_2, \\ h_{(4_1 4_1)} &= Y^{(\mu\nu)} \int_0^\sigma (\sigma - u)^2 S_{(44\mu\nu)} du + O_2, \\ h_{(4_1 4_2)} &= Y^{(\mu\nu)} \int_0^\sigma (\sigma - u)u S_{(44\mu\nu)} du + O_2. \end{aligned} \quad (213)$$

If the curvature b of the base line is small (as it is in physical applications), then, by (210), ζ is small, *provided the FC $X_{(\alpha)}$ are not large*. We can then simplify (211) a little by dropping ζ^2 and the product of ζ with an h -term.

Having thus found, to the approximation indicated, the metric for FC, let us turn to OC for which we still have (205) and (206), but for which we have, instead of (207),

$$k^{-1}U_i A^i = r > 0, \quad r^2 = X_{(\alpha)}X^{(\alpha)}, \quad (214)$$

by (193). The positive sign for r is due to the fact that we take the tangent vector A^i pointing into the future and the vector U^i pointing into the past (Fig. 12). Now (206) gives

$$W_{(\alpha)} = DX_{(\alpha)} - r b B_{(\alpha)}. \quad (215)$$

In OC, U^i is a null vector, and so

$$U_i U^i = 0, \quad U_i W^i = 0, \\ U_{(4)} W^{(4)} = - U_{(\alpha)} W^{(\alpha)} = - k X_{(\alpha)} W^{(\alpha)} = - k X_{(\alpha)} DX^{(\alpha)} + k r \zeta, \quad (216)$$

where ζ is as in (210). But

$$U_{(4)} = - U^{(4)} = (U_{(\alpha)} U^{(\alpha)})^{\frac{1}{2}} = k r, \quad (217)$$

and so

$$W^{(4)} = - r^{-1} X_{(\alpha)} DX^{(\alpha)} + \zeta = \zeta - Dr. \quad (218)$$

Hence by (198) and (203)

$$Z_{(\alpha)} = Z^{(\alpha)} = DX^{(\alpha)} - r b B^{(\alpha)}, \quad Z^{(4)} = - Z_{(4)} = 1 + \zeta - Dr. \quad (219)$$

We are now in a position to write out the metric form for optical coordinates. But, as the explicit formula is somewhat involved, we shall not complete the substitutions. It is enough to say that this form is, by (204),

$$\Phi = g_{(rs)} dX^{(r)} dX^{(s)} = 2\Omega(P_2 Q_2) \\ = ds^2 [Z_{(a)} Z^{(a)} + 2Z^{(a)} h_{(a_1 a_1)} + 2Z^{(a)} h_{(a_1 b_2)} Z^{(b)}], \quad (220)$$

wherein we are to substitute

$$Z^{(\alpha)} ds = Z_{(\alpha)} ds = dX^{(\alpha)} - r b B^{(\alpha)} dX^{(4)}, \\ Z^{(4)} ds = - Z_{(4)} ds = (1 + \zeta) dX^{(4)} - dr. \quad (221)$$

As for the h -terms in (220), we are to substitute for them, as in (97),

$$\begin{aligned} h_{(m_1 n_1)} &= \frac{3}{2}\sigma^{-1}U(r)U(s)\int_0^\sigma (\sigma - u)^2 S_{(m n r s)} du + O_2, \\ h_{(m_1 n_2)} &= \frac{3}{2}\sigma^{-1}U(r)U(s)\int_0^\sigma (\sigma - u)u S_{(m n r s)} du + O_2, \\ U^{(\alpha)} &= \sigma^{-1}X^{(\alpha)}, \quad U^{(4)} = -r\sigma^{-1}, \quad r = (X_{(\alpha)}X^{(\alpha)})^{\frac{1}{2}}. \end{aligned} \quad (222)$$

As in the case of FC, we can simplify the metric if the curvature b of the base line is small, provided that $X^{(\alpha)}$ are not large.

§ 12. GEODESICS IN TERMS OF FERMI COORDINATES AND OPTICAL COORDINATES

We shall now discuss the equations of geodesics in terms of Fermi coordinates (FC) and optical coordinates (OC). This work has a direct physical interpretation, as will be made clear later: the base line C of the coordinates is the world-line of an observer (perhaps a terrestrial astronomer) and the geodesic I' of which we are to find the equations is the world-line of some free particle — perhaps a planet.

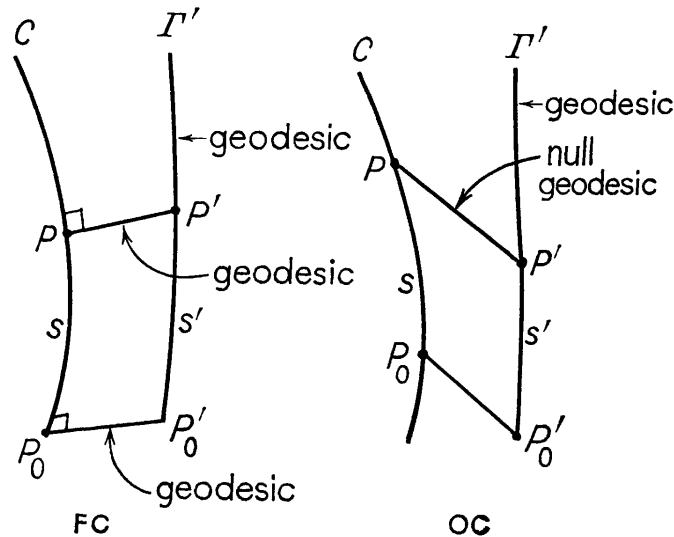


Fig. 14 – The geodesic problem for Fermi coordinates and optical coordinates

In the preceding section we found approximate expressions for the metric tensor in FC and in OC, and from that we could undoubtedly worry out the equations of a geodesic by 1-(31). However this would involve differentiating the metric tensor and it is better to start afresh. Fig. 14 shows the timelike base line C and a timelike geodesic

I' , with the correspondences (P, P') appropriate to FC and OC respectively. P_0 and P'_0 are corresponding base points on C and I' , and we write $P_0P = s$, $P'_0P' = s'$. Then $s = X^{(4)}$, the fourth coordinate in FC or in OC.

With $\Omega = \Omega(PP')$ throughout, we have

$$X^{(\alpha)} = X_{(\alpha)} = -\Omega_i \lambda^i_{(\alpha)}, \quad X^{(4)} = -X_{(4)} = s, \quad (223)$$

and, by (207) and (214),

$$\begin{aligned} \Omega_i A^i &= -\theta r, \\ \theta &= 0 \text{ for FC,} \quad \theta = 1 \text{ for OC.} \end{aligned} \quad (224)$$

By introducing this factor θ , we are able to discuss FC and OC in a single argument.

Since a geodesic is determined by a point on it and a direction, the differential equations of I' are necessarily of the form

$$D^2 X_{(\alpha)} = f_{(\alpha)}(X_{(\beta)}, DX_{(\gamma)}, s), \quad D = d/ds; \quad (225)$$

our object is to find the functions $f_{(\alpha)}$.

Let us first dispose of some preliminaries. By (180) we have ¹, with $D = \delta/\delta s$,

$$D\lambda^i_{(\alpha)} = bA^i B_{(\alpha)}, \quad \lambda^i_{(4)} = A^i, \quad (226)$$

where b is the first curvature of C . If $A^{i'}$ is the unit tangent to I' at P' , we have, since the unit tangent to a geodesic undergoes parallel transport,

$$DA^{i'} = 0, \quad A^{i'} A_{i'} = -1. \quad (227)$$

It is convenient to define

$$H^{i'} = A^{i'} Ds', \quad (228)$$

so that

$$(Ds')^2 = -H^{i'} H_{i'} \quad (229)$$

and

$$DH^{i'} = A^{i'} D^2 s' = \chi H^{i'}, \quad \chi = D^2 s' / Ds'. \quad (230)$$

Further, we define

$$L_{(a)} = \Omega_{ij} \lambda^i_{(a)} H^{j'}, \quad (231)$$

¹ This notation is consistent with that of (225), for $\delta/\delta s = d/ds$ when it operates on an invariant.

so that

$$DL_{(\alpha)} = bL_{(4)}B_{(\alpha)} + \chi L_{(\alpha)} + \Omega_{ij'k}\lambda_{(\alpha)}^i H^{j'} A^k + \Omega_{ij'k'}\lambda_{(\alpha)}^i H^{j'} H^{k'}, \quad (232)$$

$$DL_{(4)} = bL_{(\alpha)}B^{(\alpha)} + \chi L_{(4)} + \Omega_{ij'k}A^i H^{j'} A^k + \Omega_{ij'k'}A^i H^{j'} H^{k'}. \quad (233)$$

Differentiation of (223) gives, by (224) and (226),

$$DX_{(\alpha)} = \theta r b B_{(\alpha)} - \Omega_{ij}\lambda_{(\alpha)}^i A^j - L_{(\alpha)}, \quad (234)$$

and a second differentiation gives

$$\begin{aligned} D^2X_{(\alpha)} = & \theta D(r b B_{(\alpha)}) - \Omega_{ij}A^i A^j b B_{(\alpha)} - \Omega_{ij}\lambda_{(\alpha)}^i b B^j \\ & - \Omega_{ijk}\lambda_{(\alpha)}^i A^j A^k - \Omega_{ijk'}\lambda_{(\alpha)}^i A^j H^{k'} - DL_{(\alpha)}. \end{aligned} \quad (235)$$

We recognize the geodesic equation (225); we have now to evaluate the right hand side.

We have also the equation (224), and differentiation of it gives

$$\theta D r - \zeta + \Omega_{ij}A^i A^j + L_{(4)} = 0, \quad \zeta = bX_{(\alpha)}B^{(\alpha)}, \quad (236)$$

and

$$\begin{aligned} \theta D^2 r - D\zeta + 2\Omega_{ij}A^i b B^j + \Omega_{ijk}A^i A^j A^k \\ + \Omega_{ijk'}A^i A^j H^{k'} + DL_{(4)} = 0. \end{aligned} \quad (237)$$

So far all equations are exact, and without approximations the calculations become unmanageable. However, the general plan would be to use (237) to evaluate $DL_{(4)}$; then χ can be found from (233); then $DL_{(\alpha)}$ is given by (232); finally we would substitute in (235) to get the geodesic equations.

The situation is very much simplified by introducing the following approximations, all valid in physical situations:

- i) Space-time is nearly flat, and in consequence [cf. (95)]

$$\Omega_{ij} = g_{ij} + h_{ij}, \quad \Omega_{ij'} = -g_{ij'} + h_{ij'}, \quad (238)$$

where $g_{ij'}$ is the parallel propagator and the h -terms are small. further, any Ω with three subscripts is small.

- ii) The first curvature b of C is small and its rate of change (Db) is very small; also the second curvature c is small.
- iii) Γ' is nearly parallel ¹ to C .

If C were a geodesic and space-time flat, we would of course have

¹ In physical terms this means that the relative velocity is small, compared with the velocity of light.

$D^2X_{(\alpha)} = 0$. It is easily seen from (235) that, under the above approximations, $D^2X_{(\alpha)}$ is small, and so we need only retain principal parts in the right hand side of (235). Then the second and third terms cancel, and since $Ds' = 1$ approx., we have $H^{i'} = A^{i'}$ approx., and so (235) may be written

$$D^2X_{(\alpha)} = \theta D(r b B_{(\alpha)}) - DL_{(\alpha)} - \Omega_{ijk} \lambda_{(\alpha)}^i A^j A^k - \Omega_{ijk'} \lambda_{(\alpha)}^i A^j A^{k'}. \quad (239)$$

From (231), $L_{(\alpha)}$ is small and $L_{(4)} = 1$, approx. Also χ is small, and so (232) gives

$$DL_{(\alpha)} = bB_{(\alpha)} + \Omega_{ij'k} \lambda_{(\alpha)}^i A^{j'} A^k + \Omega_{ij'k'} \lambda_{(\alpha)}^i A^{j'} A^{k'}. \quad (240)$$

It is convenient to take components on the tetrad $\lambda_{(\alpha)}^i$, carried by parallel transport along PP' . Then (239) and (240) give

$$D^2X_{(\alpha)} = \theta D(r b B_{(\alpha)}) - bB_{(\alpha)} - M_{(\alpha)} \quad (241)$$

where

$$M_{(\alpha)} = \Omega_{(\alpha 44)} + 2\Omega_{(\alpha 44')} + \Omega_{(\alpha 4'4')}. \quad (242)$$

It is easy to see that we may omit the θ -term in (241). For b is small, Db is negligible, Dr is small, and, by the Frenet-Serret formulae I-(55),

$$B_{(\alpha)} = B_i \lambda_{(\alpha)}^i, \quad DB_{(\alpha)} = (cC_i + bA_i) \lambda_{(\alpha)}^i + B_i A^i b B_{(\alpha)}. \quad (243)$$

Thus, for both FC and OC, we have the following approximate equation for the geodesic I'' :

$$D^2X_{(\alpha)} = -bB_{(\alpha)} - M_{(\alpha)}. \quad (244)$$

We see here two departures from the zero value. The first is due to the curvature of the base line ¹, and the second is due to the curvature of space-time.

It remains to evaluate $M_{(\alpha)}$ and this we do by (115), making the following changes in notation:

$$u_1 = 0, \quad u_2 = \sigma, \quad k = \sigma^{-1}. \quad (245)$$

As for the U -terms in (115), we are to take $U^i = dx^i/du$ along the geodesic PP' . Thus

$$\begin{aligned} \Omega_i &= -\sigma U_i, & U^{(\alpha)} &= U_{(\alpha)} = -\sigma^{-1} \Omega_i \lambda_{(\alpha)}^i = \sigma^{-1} X_{(\alpha)}, \\ U^{(4)} &= -U_{(4)} = \sigma^{-1} \Omega_i \lambda_{(4)}^i = \sigma^{-1} \Omega_i A^i = -\theta r \sigma^{-1}, \end{aligned} \quad (246)$$

¹ This is the reason why bodies fall to the ground! This will be treated in detail later. See III-§ 9.

by (224). Note that we include here both FC ($\theta = 0$) and OC ($\theta = 1$). The value of $M_{(\alpha)}$ in (244) is then given by (242), with the following values:

$$\begin{aligned}\Omega_{(\alpha 44)} &= -3\sigma^{-2}U^{(q)} \int_0^\sigma (\sigma - u)^2 S_{(\alpha 44 q)} du \\ &\quad + \frac{3}{2}\sigma^{-2}U^{(p)}U^{(q)} \int_0^\sigma (\sigma - u)^3 S_{(\alpha 4 p q 4)} du, \\ \Omega_{(\alpha 44')} &= 3\sigma^{-2}U^{(q)} \int_0^\sigma (\sigma - u)^2 S_{(\alpha 44 q)} du \\ &\quad + \frac{3}{2}\sigma^{-2}U^{(p)}U^{(q)} \int_0^\sigma (\sigma - u)^2 u S_{(\alpha 4 p q 4)} du, \\ \Omega_{(\alpha 4' 4')} &= -3\sigma^{-2}U^{(q)} \int_0^\sigma u^2 S_{(44 \alpha q)} du \\ &\quad + \frac{3}{2}\sigma^{-2}U^{(p)}U^{(q)} \int_0^\sigma (\sigma - u) u^2 S_{(44 p q \alpha)} du.\end{aligned}\quad (247)$$

Note the change in order of the subscripts in the last formula.

As a check on these rather complicated formulae, consider the case where C is a geodesic (so that $b = 0$) and I'' is close to C . Then the principal parts in (247) are given by the first integrals, and we have, with the S -terms calculated at P ,

$$\Omega_{(\alpha 44)} = -3\sigma^{-2} \cdot \sigma^{-1} X^{(\beta)} S_{(\alpha 44 \beta)} \int_0^\sigma (\sigma - u)^2 du = -S_{(\alpha 44 \beta)} X^{(\beta)}, \quad (248)$$

$$\Omega_{(\alpha 44')} = S_{(\alpha 44 \beta)} X^{(\beta)}, \quad \Omega_{(\alpha 4' 4')} = -S_{(44 \alpha \beta)} X^{(\beta)}.$$

Then

$$M_{(\alpha)} = (S_{(\alpha 44 \beta)} - S_{(44 \alpha \beta)}) X^{(\beta)} = R_{(\alpha 4 \beta 4)} X^{(\beta)}, \quad (249)$$

and the geodesic equation (244) reads

$$D^2 X_{(\alpha)} = -R_{(\alpha 4 \beta 4)} X^{(\beta)}. \quad (250)$$

This agrees, as it should, with the equation of geodesic deviation I-(140).

§ 13. THE WORLD-FUNCTION AND ITS DERIVATIVES FOR TWO POINTS ON A TIMELIKE CURVE

Let C be a timelike curve ¹ in space-time (Fig. 15). This curve may be defined by assigning the principal tetrad (tangent and normals) at a

¹ Although as a matter of policy we keep the argument purely geometrical, this curve C might be the world-line of you or me — terrestrial observers carried on the rotating earth. As we shall see later, the results of the present calculations have simple and rather fundamental physical interpretations. See III-§ 10.

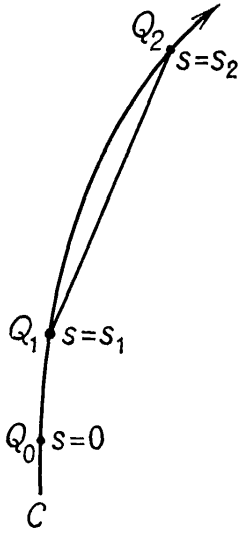


Fig. 15 - The world-function $\Omega(Q_1Q_2)$ for two points on a timelike curve

point $Q_0(s=0)$ on it, and giving the three curvatures as functions of s . Or it may be defined in the neighbourhood of the point Q_0 by assigning the unit tangent A^i and the absolute derivatives DA^i , D^2A^i , \dots at Q_0 ($D = \partial/\partial s$). We shall adopt the latter specification.

Let $Q_1 (s = s_1)$ and $Q_2 (s = s_2)$ be two points of C near Q_0 , so that s_1 and s_2 are small (O_1). The world-function $\Omega(Q_1Q_2)$ is a function of s_1, s_2 , and may be expanded in a double power series of the form

$$\begin{aligned} \Omega(Q_1Q_2) = & [\Omega] + s_1[D_1\Omega] + s_2[D_2\Omega] \\ & + \frac{1}{2}\{s_1^2[D_1^2\Omega] + 2s_1s_2[D_1D_2\Omega] + s_2^2[D_2^2\Omega]\} + \dots, \end{aligned} \quad (251)$$

where $D_1 = \partial/\partial s_1$, $D_2 = \partial/\partial s_2$, and $[]$ indicates evaluation at Q_0 where $s_1 = s_2 = 0$; these are in fact coincidence limits in the sense of Sect. 2, and may

be evaluated by (69) after a little manipulation.

We have

$$\begin{aligned} D_1\Omega &= \Omega_{i_1}A^{i_1}, \\ D_1^2\Omega &= \Omega_{i_1}D_1A^{i_1} + \Omega_{i_1j_1}A^{i_1}A^{j_1}, \\ D_1D_2\Omega &= \Omega_{i_1j_2}A^{i_1}A^{j_2}, \\ D_1^3\Omega &= \Omega_{i_1}D_1^2A^{i_1} + 3\Omega_{i_1j_1}DA^{i_1}A^{j_1} + \Omega_{i_1j_1k_1}A^{i_1}A^{j_1}A^{k_1}, \\ D_1^2D_2\Omega &= \Omega_{i_1j_2}D_1A^{i_1}A^{j_2} + \Omega_{i_1j_1k_2}A^{i_1}A^{j_1}A^{k_2}, \end{aligned} \quad (252)$$

with similar equations obtained by interchange of the numbers 1 and 2. Hence, by (69),

$$\begin{aligned} [\Omega] &= 0, \\ [D_1\Omega] &= [D_2\Omega] = 0, \\ [D_1^2\Omega] &= -1, \quad [D_1D_2\Omega] = 1, \quad [D_2^2\Omega] = -1, \\ [D_1^3\Omega] &= [D_1^2D_2\Omega] = [D_1D_2^2\Omega] = [D_2^3\Omega] = 0. \end{aligned} \quad (253)$$

The last two lines in (252) give, if we use (69) after differentiating,

$$\begin{aligned} [D_1^4\Omega] &= [4\Omega_{i_1j_1}D_1^2A^{i_1}A^{j_1} + 3\Omega_{i_1j_1}D_1A^{i_1}D_1A^{j_1} \\ &\quad + \Omega_{i_1j_1k_1m_1}A^{i_1}A^{j_1}A^{k_1}A^{m_1}], \\ [D_1^3D_2\Omega] &= [\Omega_{i_1j_2}D_1^2A^{i_1}A^{j_2} + \Omega_{i_1j_1k_1m_2}A^{i_1}A^{j_1}A^{k_1}A^{m_2}], \\ [D_1^2D_2^2\Omega] &= [\Omega_{i_1j_2}D_1A^{i_1}D_2A^{j_2} + \Omega_{i_1j_1k_2m_2}A^{i_1}A^{j_1}A^{k_2}A^{m_2}]. \end{aligned} \quad (254)$$

Completing the substitution from (69) we drop the numerical signs on the A 's, and then, by the skew-symmetry of the Riemann tensor, the Ω 's with four subscripts give zero. Also

$$A_i A^i = -1, \quad A_i D A^i = 0, \quad A_i D^2 A^i = -D A_i D A^i = -b^2, \quad (255)$$

where b is the first curvature of C . Hence we get

$$[D_1^4 \Omega] = -b^2, \quad [D_1^3 D_2 \Omega] = b^2, \quad [D_1^2 D_2^2 \Omega] = -b^2, \quad (256)$$

with similar equations obtained by interchange of 1 and 2. Substituting in (251) from (253) and (256), we get the following approximate expression for *the world-function for two points on C* :

$$\Omega(Q_1 Q_2) = -\frac{1}{2}(s_2 - s_1)^2 - \frac{1}{24}b^2(s_2 - s_1)^4 + O_5. \quad (257)$$

Note that, in this approximation, the curvature of C appears but the curvature of space-time does not.

If τ is the measure of the geodesic $Q_1 Q_2$ (shown in Fig. 15 as a straight line), we have

$$\Omega(Q_1 Q_2) = -\frac{1}{2}\tau^2, \quad (258)$$

and hence

$$\begin{aligned} \tau^2 &= (s_2 - s_1)^2 + \frac{1}{12}b^2(s_2 - s_1)^4 + O_5, \\ \tau &= (s_2 - s_1) + \frac{1}{24}b^2(s_2 - s_1)^3 + O_4, \end{aligned} \quad (259)$$

if $s_2 > s_1$. This is the same as the Euclidean formula for the chord (τ) of a circle of radius b^{-1} in terms of the arc $(s_2 - s_1)$, except for an important change in sign, as the result of which we have $\tau > (s_2 - s_1)$.

We proceed to evaluate the covariant derivatives of $\Omega(Q_1 Q_2)$, but we shall take only the invariant components on a Fermi triad $\lambda_{(\alpha)}^i$. These components are as follows, with others formed from these by interchange of the numbers 1 and 2:

$$\begin{aligned} \Omega_{(\alpha_1)} &= \Omega_{i_1} \lambda_{(\alpha)}^{i_1}, \\ \Omega_{(\alpha_1 \beta_1)} &= \Omega_{i_1 j_1} \lambda_{(\alpha)}^{i_1} \lambda_{(\beta)}^{j_1}, \quad \Omega_{(\alpha_1 \beta_2)} = \Omega_{i_1 j_2} \lambda_{(\alpha)}^{i_1} \lambda_{(\beta)}^{j_2}, \\ \Omega_{(\alpha_1 \beta_1 \gamma_1)} &= \Omega_{i_1 j_1 k_1} \lambda_{(\alpha)}^{i_1} \lambda_{(\beta)}^{j_1} \lambda_{(\gamma)}^{k_1}, \quad \Omega_{(\alpha_1 \beta_1 \gamma_2)} = \Omega_{i_1 j_1 k_2} \lambda_{(\alpha)}^{i_1} \lambda_{(\beta)}^{j_1} \lambda_{(\gamma)}^{k_2}, \\ \Omega_{(\alpha_1 \beta_1 \gamma_1 \delta_1)} &= \Omega_{i_1 j_1 k_1 m_1} \lambda_{(\alpha)}^{i_1} \lambda_{(\beta)}^{j_1} \lambda_{(\gamma)}^{k_1} \lambda_{(\delta)}^{m_1}, \\ \Omega_{(\alpha_1 \beta_1 \gamma_1 \delta_2)} &= \Omega_{i_1 j_1 k_1 m_2} \lambda_{(\alpha)}^{i_1} \lambda_{(\beta)}^{j_1} \lambda_{(\gamma)}^{k_1} \lambda_{(\delta)}^{m_2}, \\ \Omega_{(\alpha_1 \beta_1 \gamma_2 \delta_2)} &= \Omega_{i_1 j_1 k_2 m_2} \lambda_{(\alpha)}^{i_1} \lambda_{(\beta)}^{j_1} \lambda_{(\gamma)}^{k_2} \lambda_{(\delta)}^{m_2}. \end{aligned} \quad (260)$$

We proceed to calculate these expressions of the first, second, third and fourth orders with errors O_4 , O_3 , O_2 , O_1 , respectively, these orders of accuracy being those required in the next Section.

By the Fermi transport law I-(84) we have

$$\begin{aligned} D\lambda_{(\alpha)}^i &= A^i \lambda_{(\alpha)}^j DA_j, \\ D^2\lambda_{(\alpha)}^i &= DA^i \lambda_{(\alpha)}^j DA_j + A^i \lambda_{(\alpha)}^j D^2A_j, \end{aligned} \quad (261)$$

and hence, in an obvious notation,

$$\begin{aligned} D\lambda_{(\alpha)}^i A_i &= -\lambda_{(\alpha)}^j DA_j = -(DA)_{(\alpha)}, \\ D\lambda_{(\alpha)}^i DA_i &= 0, \\ D^2\lambda_{(\alpha)}^i A_i &= -\lambda_{(\alpha)}^j D^2A_j = -(D^2A)_{(\alpha)}, \\ D^2\lambda_{(\alpha)}^i DA_i &= DA_i DA^i \lambda_{(\alpha)}^j DA_j = b^2(DA)_{(\alpha)}. \end{aligned} \quad (262)$$

Expanding in a double power series, we have

$$\begin{aligned} \Omega_{(\alpha_1)} &= [\Omega_{(\alpha_1)}] + s_1[D_1\Omega_{(\alpha_1)}] + s_2[D_2\Omega_{(\alpha_1)}] \\ &\quad + \frac{1}{2}\{s_1^2[D_1^2\Omega_{(\alpha_1)}] + 2s_1s_2[D_1D_2\Omega_{(\alpha_1)}] + s_2^2[D_2^2\Omega_{(\alpha_1)}]\} + \dots \end{aligned} \quad (263)$$

Now

$$\begin{aligned} D_1\Omega_{(\alpha_1)} &= \Omega_{i_1} D_1\lambda_{(\alpha)}^{i_1} + \Omega_{i_1j_1}\lambda_{(\alpha)}^{i_1} A^{j_1}, \\ D_2\Omega_{(\alpha_1)} &= \Omega_{i_1j_2}\lambda_{(\alpha)}^{i_1} A^{j_2}, \\ D_1^2\Omega_{(\alpha_1)} &= \Omega_{i_1} D_1^2\lambda_{(\alpha)}^{i_1} + 2\Omega_{i_1j_1} D_1\lambda_{(\alpha)}^{i_1} A^{j_1} + \Omega_{i_1j_1}\lambda_{(\alpha)}^{i_1} D_1A^{j_1} \\ &\quad + \Omega_{i_1j_1k_1}\lambda_{(\alpha)}^{i_1} A^{j_1} A^{k_1}, \\ D_1D_2\Omega_{(\alpha_1)} &= \Omega_{i_1j_2} D_1\lambda_{(\alpha)}^{i_1} A^{j_2} + \Omega_{i_1j_1k_2}\lambda_{(\alpha)}^{i_1} A^{j_1} A^{k_2}, \\ D_2^2\Omega_{(\alpha_1)} &= \Omega_{i_1j_2}\lambda_{(\alpha)}^{i_1} D_2A^{j_2} + \Omega_{i_1j_2k_2}\lambda_{(\alpha)}^{i_1} A^{j_2} A^{k_2}. \end{aligned} \quad (264)$$

The coincidence limits at Q_0 are as follows:

$$\begin{aligned} [\Omega_{(\alpha_1)}] &= [D_1\Omega_{(\alpha_1)}] = [D_2\Omega_{(\alpha_1)}] = 0, \\ [D_1^2\Omega_{(\alpha_1)}] &= -[D_1D_2\Omega_{(\alpha_1)}] = [D_2^2\Omega_{(\alpha_1)}] = -(DA)_{(\alpha)}. \end{aligned} \quad (265)$$

Since our approximation is to exclude O_4 terms, we do not have to write out completely the next derivatives obtained from (264). We can proceed to the required coincidence limits partly in our heads:

thus, by (69) and (262),

$$\begin{aligned}
 [D_1^3 \Omega_{(\alpha_1)}] &= [3\Omega_{i_1 j_1} D_1^2 \lambda_{(\alpha)}^{i_1} A^{j_1} + 3\Omega_{i_1 j_1} D_1 \lambda_{(\alpha)}^{i_1} D_1 A^{j_1} + \Omega_{i_1 j_1} \lambda_{(\alpha)}^{i_1} D_1^2 A^{j_1}] \\
 &= -3(D^2 A)_{(\alpha)} + 0 + (D^2 A)_{(\alpha)} = -2(D^2 A)_{(\alpha)}, \\
 [D_1^2 D_2 \Omega_{(\alpha_1)}] &= [\Omega_{i_1 j_2} D_1^2 \lambda_{(\alpha)}^{i_1} A^{j_2}] = (D^2 A)_{(\alpha)}, \\
 [D_1 D_2^2 \Omega_{(\alpha_1)}] &= [\Omega_{i_1 j_2} D_1 \lambda_{(\alpha)}^{i_1} D_2 A^{j_2}] = 0, \\
 [D_2^3 \Omega_{(\alpha_1)}] &= [\Omega_{i_1 j_2} \lambda_{(\alpha)}^{i_1} D_2^2 A^{j_2}] = -(D^2 A)_{(\alpha)}.
 \end{aligned} \tag{266}$$

Substitution from (265) and (266) in the series (263) gives

$$\Omega_{(\alpha_1)} = -\frac{1}{2}(s_2 - s_1)^2 \{(DA)_{(\alpha)} + \frac{1}{3}(2s_1 + s_2)(D^2 A)_{(\alpha)}\} + O_4. \tag{267}$$

We have still to calculate the derivatives of the second, third, and fourth orders in (260). The calculations follow the same plan as above, but are simpler because, with increase of order, we are more tolerant in the order of approximation. The reader should find no difficulty in verifying the following formulae, which include for ease of reference (257) and (267):

$$\Omega(Q_1 Q_2) = -\frac{1}{2}(s_2 - s_1)^2 - \frac{1}{24}b^2(s_2 - s_1)^4 + O_5, \tag{268}$$

$$\begin{aligned}
 \Omega_{(\alpha_1)} &= -\frac{1}{2}(s_2 - s_1)^2 \{(DA)_{(\alpha)} + \frac{1}{3}(2s_1 + s_2)(D^2 A)_{(\alpha)}\} + O_4, \\
 \Omega_{(\alpha_2)} &= -\frac{1}{2}(s_2 - s_1)^2 \{(DA)_{(\alpha)} + \frac{1}{3}(s_1 + 2s_2)(D^2 A)_{(\alpha)}\} + O_4,
 \end{aligned} \tag{269}$$

$$\begin{aligned}
 \Omega_{(\alpha_1 \beta_1)} &= \Omega_{(\alpha_2 \beta_2)} = \delta_{\alpha\beta} + \frac{1}{2}(s_2 - s_1)^2 S_{(\alpha\beta 44)} + O_3, \\
 \Omega_{(\alpha_1 \beta_2)} &= -\delta_{\alpha\beta} - \frac{1}{2}(s_2 - s_1)^2 \{(DA)_{(\alpha)}(DA)_{(\beta)} + S_{(\alpha 44 \beta)}\} + O_3,
 \end{aligned} \tag{270}$$

$$\begin{aligned}
 \Omega_{(\alpha_1 \beta_1 \gamma_1)} &= -(s_2 - s_1)S_{(\alpha\beta \gamma 4)} + O_2, \\
 \Omega_{(\alpha_1 \beta_1 \gamma_2)} &= \Omega_{(\alpha_2 \beta_2 \gamma_2)} = (s_2 - s_1)S_{(\alpha\beta \gamma 4)} + O_2, \\
 \Omega_{(\alpha_1 \beta_2 \gamma_2)} &= -(s_2 - s_1)S_{(\beta \gamma \alpha 4)} + O_2,
 \end{aligned} \tag{271}$$

$$\begin{aligned}
 \Omega_{(\alpha_1 \beta_1 \gamma_1 \delta_1)} &= \Omega_{(\alpha_1 \beta_1 \gamma_2 \delta_2)} = \Omega_{(\alpha_2 \beta_2 \gamma_2 \delta_2)} = S_{(\alpha\beta \gamma \delta)} + O_1, \\
 \Omega_{(\alpha_1 \beta_1 \gamma_1 \delta_2)} &= -S_{(\alpha\beta \gamma \delta)} + O_1, \\
 \Omega_{(\alpha_1 \beta_2 \gamma_2 \delta_2)} &= -S_{(\alpha \delta \beta \gamma)} + O_1.
 \end{aligned} \tag{272}$$

Note that, to this order of approximation, the first and second curvatures of C appear, but the third curvature does not. Once we get to

the derivatives of the second order in (270), the curvature of space-time appears in the form of symmetrized Riemann tensor [cf. (69)], but the derivatives of this tensor do not appear.

§ 14. THE WORLD-FUNCTION IN TERMS OF FERMI COORDINATES FOR TWO POINTS ON ADJACENT TIMELIKE CURVES

In Fig. 16 we see a timelike world-line C_0 , which we take as base line of Fermi coordinates (FC). C_1 and C_2

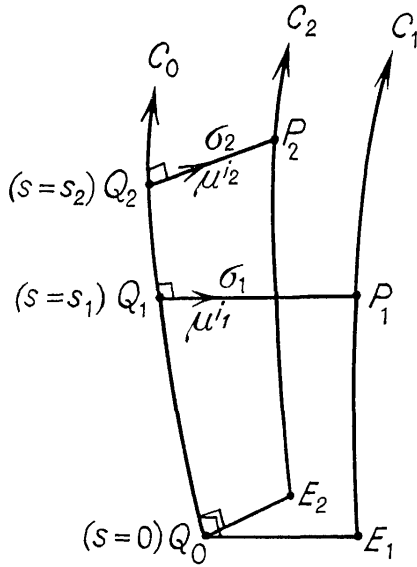


Fig. 16 – The world-function $\Omega(P_1P_2)$ in Fermi coordinates

are two other timelike world-lines adjacent to C_0 , and P_1, P_2 are points on them. Let s be the measure on C_0 , with $s = 0$ at Q_0 . Making the appropriate construction for FC, we draw the geodesics P_1Q_1 and P_2Q_2 orthogonal to C_0 . Let $s = s_1$ at Q_1 and $s = s_2$ at Q_2 ; then s_1 and s_2 are the fourth FC ($X^{(4)}$) of P_1 and P_2 respectively.

Let E_1, E_2 be the points on C_1, C_2 corresponding to $s = 0$. We regard the three-line structure C_0, C_1, C_2 as determined by the three points Q_0, E_1, E_2 and Cauchy data at these points. Our object is to calculate $\Omega(P_1P_2)$ in terms of the FC of P_1, P_2 and the Cauchy data.

Let $\lambda_{(\alpha)}^i$ be a Fermi triad on C_0 . Put $Q_1P_1 = \sigma_1$, $Q_2P_2 = \sigma_2$; let μ^{i_1} and μ^{i_2} be the unit tangents to these lines at Q_1 and Q_2 . Then the FC of P_1 and P_2 are as follows:

$$\begin{aligned} P_1: X^{(\alpha_1)} &= X_{(\alpha_1)} = \sigma_1 \mu_{i_1} \lambda_{(\alpha)}^{i_1}, & X^{(4)} &= -X_{(4)} = s_1, \\ P_2: X^{(\alpha_2)} &= X_{(\alpha_2)} = \sigma_2 \mu_{i_2} \lambda_{(\alpha)}^{i_2}, & X^{(4)} &= -X_{(4)} = s_2. \end{aligned} \quad (273)$$

If we keep the geodesics Q_1P_1, Q_2P_2 fixed, but vary σ_1, σ_2 , thus sliding P_1, P_2 in or out, $\Omega(P_1P_2)$ is a function of σ_1, σ_2 , and we can develop it in a double power series of the form

$$\Omega(P_1P_2) = \Omega(Q_1Q_2) + L_1 + \frac{1}{2}L_2 + \frac{1}{6}L_3 + \frac{1}{24}L_4 + O_5. \quad (274)$$

Here O_5 means of the fifth order in $\sigma_1, \sigma_2, s_1, s_2$, each of which is supposed to be O_1 . As for the L -terms, they are

$$\begin{aligned} L_1 &= \sigma_1(D_1\Omega) + \sigma_2(D_2\Omega), \\ L_2 &= \sigma_1^2(D_1^2\Omega) + 2\sigma_1\sigma_2(D_1D_2\Omega) + \sigma_2^2(D_2^2\Omega), \end{aligned} \quad (275)$$

and so on, where $D_1 = \partial/\partial\sigma_1$, $D_2 = \partial/\partial\sigma_2$ and $()$ means evaluation for $\sigma_1 = \sigma_2 = 0$, i.e. at Q_1, Q_2 (these are *not* coincidence limits).

Let us now give a slightly different meaning to μ^{i_1} and μ^{i_2} , taking them to be the unit tangents at P_1 and P_2 , regarded as current points on Q_1P_1 and Q_2P_2 . Then, the arguments of Ω being P_1, P_2 , we have

$$\begin{aligned} D_1\Omega &= \Omega_{i_1}\mu^{i_1}, & D_2\Omega &= \Omega_{i_2}\mu^{i_2}, \\ D_1^2\Omega &= \Omega_{i_1j_1}\mu^{i_1}\mu^{j_1}, & D_1D_2\Omega &= \Omega_{i_1j_2}\mu^{i_1}\mu^{j_2}, \end{aligned} \quad (276)$$

and so on. We proceed now to the limits $()$ and substitute in (275). The invariants then occurring can be expressed in terms of invariant components on the Fermi triad, and we get the following expressions:

$$\begin{aligned} L_1 &= \Omega_{(\alpha_1)}X^{(\alpha_1)} + \Omega_{(\alpha_2)}X^{(\alpha_2)}, \\ L_2 &= \Omega_{(\alpha_1\beta_1)}X^{(\alpha_1)}X^{(\beta_1)} + 2\Omega_{(\alpha_1\beta_2)}X^{(\alpha_1)}X^{(\beta_2)} + \Omega_{(\alpha_2\beta_2)}X^{(\alpha_2)}X^{(\beta_2)}, \\ L_3 &= \Omega_{(\alpha_1\beta_1\gamma_1)}X^{(\alpha_1)}X^{(\beta_1)}X^{(\gamma_1)} + 3\Omega_{(\alpha_1\beta_1\gamma_2)}X^{(\alpha_1)}X^{(\beta_1)}X^{(\gamma_2)} \\ &\quad + 3\Omega_{(\alpha_1\beta_2\gamma_2)}X^{(\alpha_1)}X^{(\beta_2)}X^{(\gamma_2)} + \Omega_{(\alpha_2\beta_2\gamma_2)}X^{(\alpha_2)}X^{(\beta_2)}X^{(\gamma_2)}, \\ L_4 &= \Omega_{(\alpha_1\beta_1\gamma_1\delta_1)}X^{(\alpha_1)}X^{(\beta_1)}X^{(\gamma_1)}X^{(\delta_1)} + 4\Omega_{(\alpha_1\beta_1\gamma_1\delta_2)}X^{(\alpha_1)}X^{(\beta_1)}X^{(\gamma_1)}X^{(\delta_2)} \\ &\quad + 6\Omega_{(\alpha_1\beta_1\gamma_2\delta_2)}X^{(\alpha_1)}X^{(\beta_1)}X^{(\gamma_2)}X^{(\delta_2)} \\ &\quad + 4\Omega_{(\alpha_1\beta_2\gamma_2\delta_2)}X^{(\alpha_1)}X^{(\beta_2)}X^{(\gamma_2)}X^{(\delta_2)} + \Omega_{(\alpha_2\beta_2\gamma_2\delta_2)}X^{(\alpha_2)}X^{(\beta_2)}X^{(\gamma_2)}X^{(\delta_2)}. \end{aligned} \quad (277)$$

The arguments of the Ω -terms are now Q_1, Q_2 , and in fact these terms are precisely those already evaluated in (269)–(272). When we substitute from those equations, we are near our goal; however the FC occurring in (277) are those of P_1, P_2 , and we wish to use the Cauchy data at E_1, E_2 . Using a bar to indicate evaluation at E_1, E_2 , we make the expansions

$$\begin{aligned} X^{(\alpha_1)} &= \bar{X}^{(\alpha_1)} + s_1\bar{D}\bar{X}^{(\alpha_1)} + \frac{1}{2}s_1^2\bar{D}^2\bar{X}^{(\alpha_1)} + O_3, \\ X^{(\alpha_2)} &= \bar{X}^{(\alpha_2)} + s_2\bar{D}\bar{X}^{(\alpha_2)} + \frac{1}{2}s_2^2\bar{D}^2\bar{X}^{(\alpha_2)} + O_3, \end{aligned} \quad (278)$$

where $D = d/ds$ (note that the differentiation is with respect to the measure of C_0 , i.e. the fourth Fermi coordinate, and not with respect to the measures of C_1 and C_2). We then substitute these expansions in (277) and use (269)–(272). To avoid burdening the notation, we shall delete the bars, but evaluation of the Fermi coordinates and their

derivatives at E_1, E_2 is understood. The derivatives of A are curvature properties of C_0 at Q_0 , and the components of the symmetrized Riemann tensor (S -terms) are taken at Q_0 . The result is as follows:

$$\Omega(P_1P_2) = M_2 + M_3 + N_3 + M_4 + N_4 + O_5, \quad (279)$$

where

$$M_2 = -\frac{1}{2}(s_2 - s_1)^2 + \frac{1}{2}r_{12}^2, \quad r_{12}^2 = (X_{(\alpha_1)} - X_{(\alpha_2)})(X^{(\alpha_1)} - X^{(\alpha_2)}), \quad (280)$$

$$M_3 = -\frac{1}{2}(s_2 - s_1)^2(X^{(\alpha_1)} + X^{(\alpha_2)})(DA)_{(\alpha)}, \quad (281)$$

$$N_3 = (X_{(\alpha_1)} - X_{(\alpha_2)})(s_1DX^{(\alpha_1)} - s_2DX^{(\alpha_2)}), \quad (282)$$

$$\begin{aligned} M_4 = & -\frac{1}{2}(s_2 - s_1)^2S_{(\alpha_4\alpha_4\beta)}(X^{(\alpha_1)}X^{(\beta_1)} + X^{(\alpha_1)}X^{(\beta_2)} + X^{(\alpha_2)}X^{(\beta_2)}) \\ & + \frac{1}{2}(s_2 - s_1)S_{(\alpha\beta\gamma_4)}(X^{(\alpha_1)}X^{(\beta_1)}X^{(\gamma_2)} - X^{(\alpha_2)}X^{(\beta_2)}X^{(\gamma_1)}) \\ & + \frac{1}{4}S_{(\alpha\beta\gamma\delta)}X^{(\alpha_1)}X^{(\beta_1)}X^{(\gamma_2)}X^{(\delta_2)}, \end{aligned} \quad (283)$$

$$\begin{aligned} N_4 = & -\frac{1}{24}b^2(s_2 - s_1)^4 - \frac{1}{2}(s_2 - s_1)^2(s_1DX^{(\alpha_1)} + s_2DX^{(\alpha_2)})(DA)_{(\alpha)} \\ & - \frac{1}{2}(s_2 - s_1)^2(DA)_{(\alpha)}(DA)_{(\beta)}X^{(\alpha_1)}X^{(\beta_2)} \\ & - \frac{1}{6}(s_2 - s_1)^2\{(2s_1 + s_2)X^{(\alpha_1)} + (s_1 + 2s_2)X^{(\alpha_2)}\}(D^2A)_{(\alpha)} \\ & + \frac{1}{2}(X_{(\alpha_1)} - X_{(\alpha_2)})(s_1^2D^2X^{(\alpha_1)} - s_2^2D^2X^{(\alpha_2)}) \\ & + \frac{1}{2}(s_1^2DX^{(\alpha_1)}DX_{(\alpha_1)} - 2s_1s_2DX^{(\alpha_1)}DX_{(\alpha_2)} + s_2^2DX^{(\alpha_2)}DX_{(\alpha_2)}) \end{aligned} \quad (284)$$

In the above calculation, use was made of the identity

$$S_{(\alpha\beta 44)} = -2S_{(\alpha 44\beta)}. \quad (285)$$

As a check on (279), we note that it should be, and is, invariant under interchange of the numbers 1 and 2.

This work represents a formal calculation in which $\sigma_1, \sigma_2, s_1, s_2$ are treated as small quantities of the first order, the subscripts on M and N in (279) indicating orders of magnitude. We have not made any assumption of smallness regarding the curvature of space-time or the curvatures of the three world-lines, nor have we assumed these world-lines to be nearly parallel. The general effect of such additional assumptions is to reduce the dimensionless ratios $M_3/M_2, N_3/M_2$, etc. The curvature of space-time appears only in M_4 , and flatness implies $M_4 = 0$. On the other hand, if C_0 is a geodesic and the Fermi coordinates are constants, then $M_3 = N_3 = N_4 = 0$. In applying (279) to a physical situation in XI-§ 7, we shall drop N_4 but keep M_4 in order to explore the effect of the gravitational field.

CHAPTER III

CHRONOMETRY IN RIEMANNIAN SPACE-TIME

§ 1. NATURAL OBSERVATIONS (NO) AND MATHEMATICAL OBSERVATIONS (MO)

Except for occasional hints as to possible physical applications, the two preceding chapters consisted of pure mathematics (Riemannian geometry). The mathematical argument may have lacked complete logical rigour, but it is safe to say that those chapters contained no material for controversy; mathematics is not a controversial subject because all mathematicians attach the same meanings to the terms used in it.

Neither is experimental physics a controversial subject. But theoretical physics is, and always will be. This is inevitable, since the aim of theoretical physics is to force the vast complexity of nature into a narrow mathematical mould, using idealizations and simplifications which are absolutely necessary and (to the unsympathetic mind) absolutely nonsensical.

Since physics is based on observations, it is useful to list four types of observation ¹ as follows:

- | | |
|---|--------|
| (i) Uncontrolled natural observations (UNO) | } (NO) |
| (ii) Controlled natural observations (CNO) | |
| (iii) Imagined natural observations (INO) | |
| (iv) Mathematical observations (MO). | |

The meanings here attached to these terms are best explained by examples. UNO are performed by astronomers observing stars or photographing the heavens, and by meteorologists weighing raindrops. CNO are performed by physicists timing a simple pendulum or observing the scatter of a beam of protons issuing from an accelerator. INO are performed by physicists planning experiments in advance, and by

¹ Neither *observation* nor *experiment* is a wholly satisfactory word. For simplicity and uniformity, the word *observation* alone is used, although in some contexts *experiment* would be better.

astronomers discussing the rotundity of the moon's backside¹ or the position of Jupiter a century hence. However, the divisions between UNO, CNO and INO are not sharp. UNO today may be CNO tomorrow, and vice versa. Every observation is INO before it is performed, UNO or CNO afterwards. It may be better to forget the differences and lump UNO, CNO and INO together under the sign NO.

Between NO and MO there is a sharp and decisive break. Only the simplest MO (counting) can be regarded as being NO also (e.g. the observation that 23 is a prime number). Generally MO involve infinity (irrational numbers, differential calculus, and so on), and so lie outside physics and outside nature (except in so far as the mind of man is natural). As examples of MO we have the concurrence of the angular bisectors in Euclidean geometry, or the ellipticity of planetary orbits in Newtonian astronomy. This concurrence and this ellipticity are meaningless in terms of NO, for in nature there are neither straight lines nor ellipses.

The peculiar fascination of theoretical physics lies in the art of forcing meaningful truth out of the meaningless equation $NO = MO$, which is a symbolic form of the assertion that natural phenomena obey exact mathematical laws. The true inequality $NO \neq MO$ should not be spoken above a whisper, because it is extremely dangerous. If believed, it would sever mathematics from physics, and reduce both to sterility through lack of mutual fecundation. It is whispered here only as an apology to those readers who expect to see the mathematics of relativity tied to the physics of relativity by a strong chain of clear thought. It cannot be done. We have to muddle through. And if this book is dishonest in confusing MO with NO, it is no more dishonest than all similar books are, and necessarily must be. This sad state of affairs is not peculiar to relativity; every branch of mathematical physics has in its cupboard the skeleton $MO \neq NO$.

The preceding remarks are no more than the briefest introduction to a subject on which whole books might be written and in which the confusion is necessarily so great that any author might well be satisfied if he succeeded in reducing the confusion by ever so little — there is no hope of abolishing it. Let us, however, realize the presence of this difficulty, this confusion, because only through this realization may we hope to be restrained from controversies full of sound and fury, signifying nothing. Some the difficulties of the situation are treated

¹ Since this was written, INO became UNO.

admirably by BRIDGMAN [1949] in an article discussing the value and the limitations of the so-called *operational method*, in which the meaning of a term is to be sought in the operations employed in making application of the term. It was by use of this method that Einstein gave a convincing argument against the absolute time of Newton, and Bridgman accuses Einstein of failing to carry the method over into general relativity. The present book is an attempt to make general relativity more operational than it has been hitherto, but I have no false confidence in the complete clarity of the method, and as the argument progresses it will become increasingly obscure to the author and to the reader whether the 'experiments' considered (sometimes called *ideal experiments* or *thought experiments*) belong to the class MO or to the class NO. This matter will be referred to again briefly in § 3, and after that we shall forget it.

§ 2. CHRONOMETRY AND THE RIEMANNIAN HYPOTHESIS

The key-word in relativity is *event*¹. The totality of all possible events form a 4-dimensional continuum, called *space-time*, in which coordinates x^i may be taken in a great variety of ways.

We accept the concept of a *material particle*. Its history is a sequence of events — a curve in space-time; we call it a *world-line*.

We accept the concept of *time-order* on the world-line of any material particle, so that any event on that world-line separates all the other events on it into two classes — the *past* and the *future*.

Any monotonic parameter, increasing from the past into the future, might be used to measure *time* on the world-line of a material particle. However, on account of its arbitrariness, such a measure of time could have little physical significance. We make this concept of time more concrete by assuming the existence of *standard clocks*, which may be carried by material particles, and the ticking of which provides a measure of *proper time*. Since this is the only time of basic importance in relativity, we shall drop the adjective *proper* and refer to it simply as *time*².

In thus setting up, as a basic concept, the measurement of time along the world-line of any material particle, we have established a *chrono-*

¹ The word *event* and other concepts discussed here have been treated in much the same way, but in greater detail, in SYNGE [1956a], Chap. 1.

² Later we may have occasion to refer to *coordinate time*; there should be no confusion.

metry. The observation of this time is a mathematical observation (MO) — it would be foolish to inquire the colour of the eyes of the observer, because mathematical observations are made by eyeless headless mannikins. But it is by no means foolish to inquire by what apparatus a natural observation (NO) of time may be made. To deal with such questions, it is wise to be methodical and set out the replies in dictionary form. In the present case we may make the following dictionary entries, the first column showing the mathematical words and the second the appropriate corresponding physical words:

MO	NO
time	time
standard clock	atom
ticking of standard clock	emission of wave crests of radiation

To enlarge on the meaning of the NO column, we take a simple view of the radiation from an excited atom, and think of electromagnetic radiation emitted in clear-cut waves. But what atom, and what energy levels are involved in the radiation? The answer is that it does not matter, provided we use consistently the same type of atom (all atoms of the same type are regarded as identical) and the same pair of energy levels. For definiteness, we may decide to use an atom of cadmium, and the radiation which gives the red line.

It is necessary to expose here a certain physical assumption inherent in the structure of relativity. Let C (Fig. 1) be the world-line of a material particle, and B, A two events on it, with B before A . The

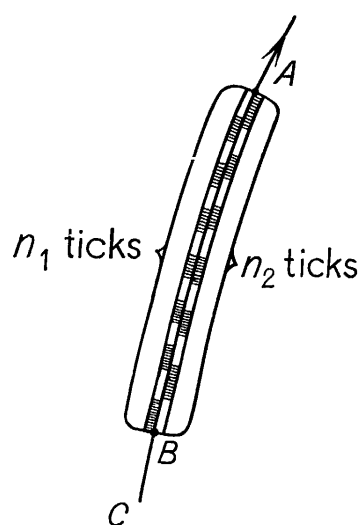


Fig. 1 — Consistency of standard clocks

particle carries two standard clocks consisting of atoms of different types, or two atoms of the same type but with the use of different energy levels. Each clock registers a definite number of ticks between B and A ; let these numbers be denoted by n_1 and n_2 . The physical assumption just referred to is the following *hypothesis of consistency*: For two standard clocks, the ratio $n_1 : n_2$ is a natural constant, independent of the world-line on which the observations are made and of the events on that world-line.

The modern mental picture of an atom and the radiation from it is so unclear

that it would be idle to ask for a straight definite answer to the question whether the hypothesis of consistency is true or false. The most we can do is to state that there is no evidence against it as a sharpened concept (MO) from spectroscopy (NO), with neglect of line-breadth, and we shall accept it in setting up the physics of general relativity. If we did not, we would either have to abandon general relativity in its present form, or else substitute a strange assumption that, out of all possible atomic clocks, there was one (or perhaps a privileged class) by which alone we could measure a time which was of physical importance. On the other hand, armed with the hypothesis of consistency, we can reiterate what was written above — the particular standard clock used does not matter, because the only effect of changing from one clock to another is to change the unit of time, the ratio of two units being a universal constant. It is important to note, however, that nature does not here prescribe a natural unit of time, and in setting up the equations of relativity this should be borne in mind. Apart from this caution, we shall for simplicity suppose that some definite standard clock has been selected once for all — it might well be a cadmium atom emitting the red line. Since all clocks henceforth considered are standard clocks, we shall drop the adjective and call them simply *clocks*.

We come now to the essence of general relativity — the chronometric assumption which makes space-time Riemannian. Let x^i and $x^i + dx^i$ be the coordinates of two adjacent events in the history of a material particle. Let ds be the corresponding time registered by a clock carried by the particle. Then ds is a function of x^i and dx^i , necessarily homogeneous of the first degree in the differentials. We make the *Riemannian hypothesis*:

$$ds^2 = - g_{ij} dx^i dx^j, \quad (1)$$

where g_{ij} ($= g_{ji}$) are functions of x^i . The tensorial character of g_{ij} follows from the invariance of ds (it has been defined without reference to any particular coordinate system). We further assume that the quadratic form

$$\Phi = g_{ij} dx^i dx^j \quad (2)$$

is of signature $+2$, as in I-§ 1. To complete the mathematical tie-up with Chap. I, we assume the existence of admissible coordinates in space-time, for which, it will be recalled, g_{ij} and their first derivatives are continuous.

We have now invested the quadratic form Φ with physical meaning, for $ds = (-\Phi)^{\frac{1}{2}}$ and this can be measured by the clock carried by the particle. But this works only if Φ is negative; if Φ is positive, (1) makes ds imaginary.

There has been a good deal of confusion in relativity theory concerning the physical meaning of Φ , a confusion which reflects the confused and semi-mystical attitude of mathematicians to geometry prior to the time of Hilbert. It seems to have been thought that Φ has two physical interpretations of entirely different kinds according as Φ is negative or positive. If Φ is negative, we have the chronometric interpretation precisely as set out above. But if Φ is positive, it has been customary to regard it as a measure of *length*. That is not the procedure used in this book. *For us time is the only basic measure*. Length (or distance), in so far as it is necessary or desirable to introduce it, is strictly a derived concept, and will be dealt with later in that spirit.

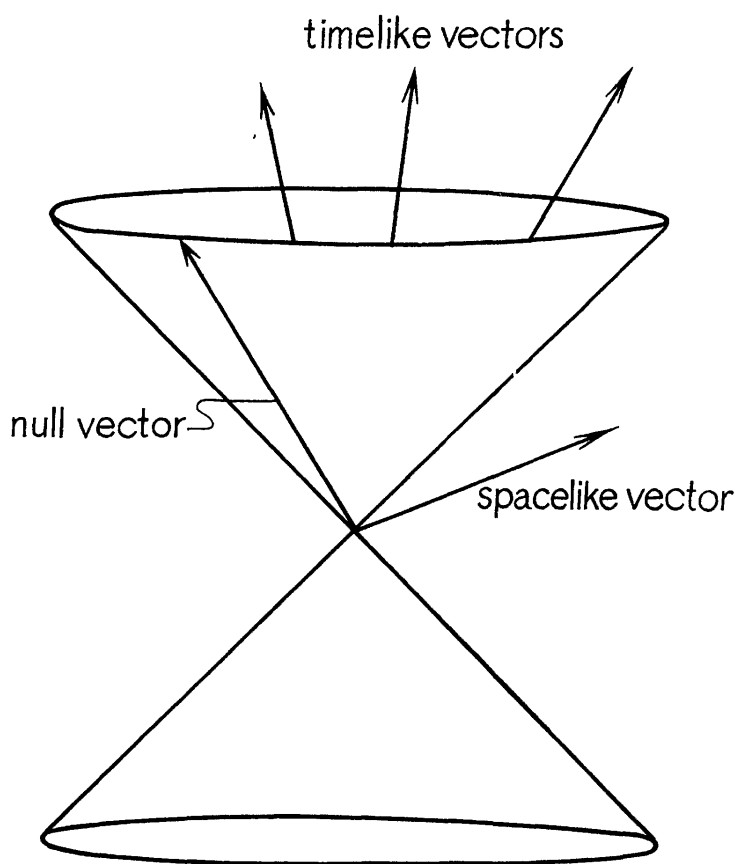


Fig. 2 – Elementary null cone, with timelike, null, and spacelike vectors

We are now launched on the task of giving physical meaning to the Riemannian geometry of Chaps. I and II. It is indeed a Riemannian *chronometry* rather than *geometry*, and the word *geometry*, with its dangerous suggestion that we should go about measuring *lengths* with

yardsticks, might well be abandoned altogether in the present connection were it not for the fact that the crude literal meaning of the word geometry has been transmuted into the abstract mathematical concept of a 'space' with a 'metric' in it.

Fig. 2 shows the elementary null cone at an event. Timelike vectors lie inside the null cone; they are physically identified as tangents to possible histories of material particles. Null vectors lie on the null cone; we identify them physically as tangents to possible histories of *photons* (particles of light), provided that, in a continuous medium, the photons are of very high energy. (Prior to Chap. XI, the word *photon* will be understood in that sense.) A spacelike vector lies outside the null cone; a curve to which it is tangent cannot be the history of a particle or photon.

But these interpretations are in the small and are common to both the special and general theories of relativity. The essence of Einstein's general theory lies in the assumption that gravitation manifests itself in the curvature of Riemannian space-time. If the Riemann tensor R_{ijkl} of the metric (1) were to vanish, we would be back in the flat space-time of gravitationless special relativity. In fact, we may write symbolically

$$R_{ijkl} = \text{gravitational field.}$$

In Chap. IV we shall discuss the field equations which connect the curvature of space-time with the matter in it, but meanwhile we shall merely assume that space-time is curved without assigning specific causes for the curvature.

§ 3. THE GEODESIC HYPOTHESIS

In Newtonian physics, a particle is said to be *free* if no force acts on it, not even the force of gravity. In relativity there is no such thing as the force of gravity, for gravity is built into the structure of space-time, and exhibits itself in the curvature of space-time, i.e. in the non-vanishing of the Riemann tensor R_{ijkl} . We recognize as forces only the effects of mechanical stresses or electromagnetic fields; with that understanding, we can say with Newton that a particle is free when no force acts on it.

We assume that a particle possesses an invariant *proper mass* m , which is constant. Since *mass* for us will always mean *proper mass*, we shall drop the useless adjective. A particle has also a world-line,

and its 4-velocity V^i is the unit tangent to its world-line, so that

$$V^i = dx^i/ds, \quad V_i V^i = -1. \quad (3)$$

The 4-momentum of the particle is

$$p^i = mV^i. \quad (4)$$

As a natural generalization of Newton's first law, we make the *geodesic hypothesis* to the effect that the world-line of a free particle is a geodesic in space-time; in symbols,

$$DV^i = 0 \quad (D = \delta/\delta u) \quad (5)$$

where u is any monotonic parameter on the world-line (perhaps the time s). From the assumed constancy of m , we deduce from (4) and (5) the equation

$$Dp^i = 0, \quad (6)$$

so that the 4-momentum of a free particle undergoes parallel transport.

The particle just considered is a material particle. A photon has zero mass ($m = 0$), but it possesses a 4-momentum p^i . We extend the geodesic hypothesis to cover photons by assuming that the world-line of a photon is a null geodesic. We add the further assumptions that p^i is tangent to the world line, and that it undergoes parallel transport along it, so that the formula (6) holds for a photon as well as for a material particle.

We saw in Chap. I that any geodesic (null or not) possesses a class of special parameters u , subject only to linear transformations, such that the equations of the geodesic read

$$D \frac{dx^i}{du} = 0 \quad (D = \delta/\delta u). \quad (7)$$

If we use one of these special parameters on the world-line of a material particle or of a photon, we have

$$p_i = \theta \frac{dx^i}{du}, \quad (8)$$

where θ is some scalar. Applying the operator D and using (6) and (7), we get

$$D\theta \cdot \frac{dx^i}{du} = 0, \quad D\theta = 0, \quad (9)$$

so that θ is constant along the world-line. In the case of a material

particle, one member of the class of special parameters is distinguished above the others, viz. $u = s$, the time. But in the case of a photon, we have $s = 0$, and the distinguished special parameter must be sought otherwise. The simplest demand is $\theta = 1$. This yields a distinguished special parameter u such that

$$p^i = \frac{dx^i}{du}. \quad (10)$$

If we apply this idea to the material particle, we get a distinguished special parameter u such that

$$du = \frac{ds}{m}. \quad (11)$$

Simple as all this may sound, and satisfactory as it may seem as MO (in the sense of § 1), we are treading on the brink of controversy. A physicist may well ask to be given samples of material particles or photons, visible to his eyes or tangible to his hands through the medium of suitable apparatus. Is the material particle perhaps the sun, the moon, a rocket, or a hydrogen atom? Is the photon a parcel of γ -rays or radio waves? If our interest were solely to construct a rational mathematical scheme labelled 'relativity', the demand for a sample might be brushed aside with as much contempt as the demand for the production of a rod of length $\sqrt{2}$ cm, for everyone knows that irrational numbers belong to MO, not NO, and our particles and photons have that artificiality too.

Later on, we shall deal with the motion of finite portions of matter and with electromagnetic waves, and it might be thought that then the difficulty of passing between MO and NO might be overcome by means of a limiting process in which the finite body shrinks to a mere point, and likewise for a concentration of an electromagnetic field. Such considerations do indeed throw light on a difficult situation, but they do not resolve the difficulty.

At this point, the best thing to say is that we are engaged in constructing a logically consistent mathematical scheme (MO) with certain physical labels attached (guides to NO), and that the practising physicist must use his judgment in the interpretation of the legends on those labels. That is, indeed, the usual procedure in all branches of theoretical physics.

§ 4. SPATIAL MEASURE, ORTHOGONALITY, AND SCALAR PRODUCTS ¹

Two adjacent events define an infinitesimal vector in space-time. If that vector is timelike, its measure or magnitude has a simple physical, or chronometric, meaning; it is the time recorded by a clock carried by a particle which includes those two events in its history. But if the vector is spacelike, this interpretation of the measure fails, because the two events cannot be included in the history of a single clock. Nevertheless an infinitesimal spacelike vector can be measured chronometrically, as we shall now show.

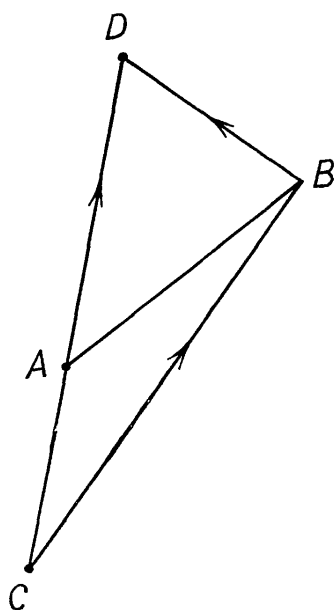


Fig. 3 – Spatial measure (or length) of AB found chronometrically (note that AB has no arrow on it!)

Fig. 3 shows an infinitesimal spacelike vector AB . Through A draw any timelike curve, and let C, D be the events where this curve intersects the null cone having B for vertex. This is a physical construction, since CAD may be taken to be the history of a particle and the null lines CB, BD to be the histories of photons. Working in the infinitesimal domain, and using an obvious notation for the infinitesimal vectors, we have

$$BD^i = AD^i - AB^i, \quad CB^i = CA^i + AB^i. \quad (12)$$

From the null, timelike and spacelike characters of the vectors involved, we have

$$BD_i BD^i = 0, \quad CB_i CB^i = 0, \quad AD_i AD^i = -AD^2, \\ CA_i CA^i = -CA^2. \quad AB_i AB^i = AB^2, \quad (13)$$

where AD, CA, AB are Riemannian measures, AD and CA being times measured by a clock carried on CAD . Also, in our infinitesimal domain,

$$CA^i = \theta AD^i, \quad CA = \theta AD, \quad (14)$$

where θ is a positive scalar. From (12) and (13) we have

$$-AD^2 - 2AD_i AB^i + AB^2 = 0, \\ -CA^2 + 2CA_i AB^i + AB^2 = 0. \quad (15)$$

¹ The curvature of space-time is not involved here, and the interpretations are common to the special and general theories; cf. SYNGE [1956a], where Chap. III in particular may help the reader who finds space-time diagrams difficult.

Eliminating the middle terms by means of (14), we get

$$AB^2 = \theta AD^2 = CA \cdot AD, \quad (16)$$

which expresses the spacelike measure AB in terms of the chronometric measures CA , AD . We call this measure the *length* of the infinitesimal spacelike vector.

If it happens that $CA = AD$, then $\theta = 1$, and we get

$$AB = CA = AD, \quad (17)$$

and from (15)

$$AD_i AB^i = 0, \quad (18)$$

which is the condition for the *orthogonality* of AB^i and AD^i . Thus we have a chronometric interpretation of the orthogonality of two vectors, one spacelike and the other timelike.

Another important orthogonality is that of two spacelike vectors. There exist ∞^1 timelike directions (as V^i in Fig. 4) orthogonal¹ to any given orthogonal spacelike pair (shown as AB , AC in Fig. 4). The condition of orthogonality of the spacelike pair is

$$AB_i AC^i = 0, \quad (19)$$

and it is easy to see (by introducing special local coordinates) that this is equivalent to the formula of Pythagoras

$$BC^2 = AB^2 + AC^2. \quad (20)$$

Since AB , BC , AC have already been given chronometric meanings, this is a chronometric equation.

The scalar product of a pair of vectors is a very important thing in physics, and we might like to have a direct physical (i.e. chronometric) interpretation of each scalar product which occurs. As part of such a programme, we see in (15) a chronometric interpretation of the scalar product of two infinitesimal vectors, one timelike and the other

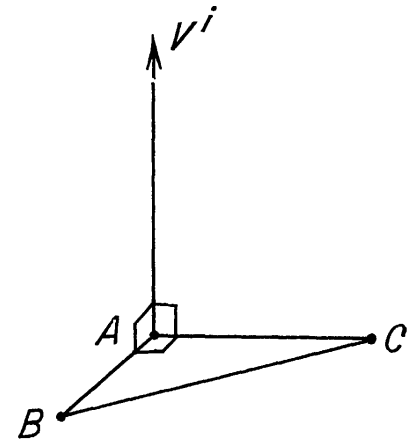


Fig. 4 – Orthogonality of two spacelike vectors, and a timelike vector orthogonal to them

¹ The 2-element defined by a pair of *non-orthogonal* spacelike directions may cut the null cone; in that case there is no timelike direction orthogonal to them both.

spacelike. But it would be tedious to develop such interpretations systematically, and it seems best to go ahead without them, noting, however, the following mathematical facts about scalar products:

- (i) If U^i and V^i are two unit vectors, both timelike and pointing into the future, then

$$U_i V^i \leq -1. \quad (21)$$

- (ii) If V^i is a fixed timelike vector, and U^i a spacelike unit vector with arbitrary direction, then $U_i V^i$ can take all real values.

- (iii) If V^i is a timelike vector and N^i a null vector, both pointing into the future, then

$$N_i V^i < 0. \quad (22)$$

In § 3 we met the 4-momentum p^i of a material particle or photon. We might refer to the first three components of p^i as 3-momentum and to the fourth component p^4 as energy, but it is wiser to reserve these important physical names for invariants, because the values of the vector components depend, of course, on the choice of coordinate system. Accordingly, we defer the definition of 3-momentum to the next Section, and here define only the *energy relative to an observer with 4-velocity V^i* by the formula

$$E = -p_i V^i. \quad (23)$$

Naturally this definition applies only if the observer and the material particle or photon have a common event in their histories, E being evaluated at that common event. The sign in (23) is chosen so that [cf. (21) and (22)] E is positive if V^i and p^i both point into the future. This is the normal situation, but we might hesitate in the case of p^i , because there are hints in modern physics that p^i might point into the past; this would mean negative energy.

§ 5. BORN RIGIDITY AND FRAMES OF REFERENCE

Consider a single infinity of timelike curves, not geodesics in general. They form a 2-space W_2 , and in W_2 we may draw the orthogonal trajectories of the given system of curves, so that we have a net, as shown in Fig. 5. Let us parametrize W_2 so that it has the equations $x^i = x^i(u, v)$, with $v = \text{const.}$ on each of the timelike curves and $u = \text{const.}$ on each of the orthogonal trajectories. As in I-(125), we write

$$\partial x^i / \partial u = U^i, \quad \partial x^i / \partial v = V^i, \quad (24)$$

and as in I-(127) we have, for the deviation vector drawn from the timelike curve (v) to the timelike curve $(v + \delta v)$,

$$\eta^i = V^i \delta v. \quad (25)$$

On account of the orthogonality in the construction, we have

$$U_i V^i = 0; \quad (26)$$

the deviation vector η^i is orthogonal to U^i everywhere on W_2 .

In general, the magnitude η of the deviation vector will vary as we pass along any one of the timelike curves. Following the definition given by Born in special relativity, we shall say that two adjacent timelike curve are *rigidly connected* if $\eta = \text{const.}$ as we pass along the curves.

Since, as shown in the preceding Section, η has a chronometric measure, rigidity may be tested by sending photons from one timelike curve to the other, and receiving back the scattered or reflected photons. The criterion for rigidity is that the elapsed time from emission to return (the *trip-time*) should be constant. This test is shown in Fig. 6, the trip-times being the chronometric measures A_1B_1, A_2B_2, \dots . There is nothing very novel about these ideas, for this test of rigidity by measuring trip-times is really the same as the testing of length by means of an interferometer, which instrument is essentially a device for comparing trip-times.

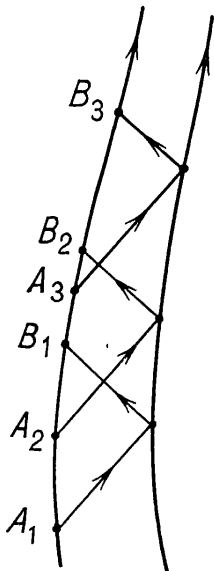


Fig. 6 - Optical (chronometric) test of Born rigidity ($A_1B_1 = A_2B_2 = \dots$)

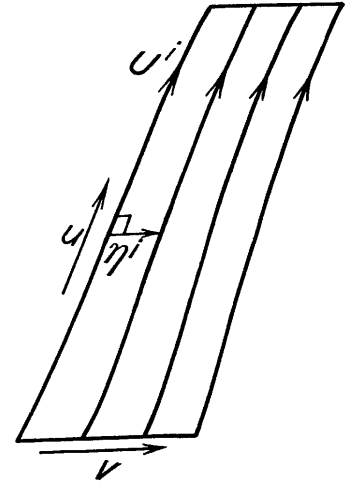


Fig. 5 - Born rigidity ($d\eta/du = 0$)

Although the Born criterion of rigidity involves no difficulty when applied to a one-dimensional body (a rod), difficulties accumulate with increase of dimensionality, and it must be stated emphatically that the three-dimensional concept of rigidity does *not* pass from Newtonian physics into relativity. We should look very sceptically at any relativistic argument which uses or implies the concept of rigidity as if the meaning of that word were clear and obvious. Thus the concept of a frame of reference (a rigid body in Newtonian physics) must be examined and redefined.

The difficulties inherent in relativistic rigidity are, however, connected with non-integrability, and are avoided if we work in an infinitesimal domain. We start with a timelike curve C_0 (Fig. 7), which we regard as given; it might be the world-line of an observer on the earth, or of his eye, or of a portion of a photographic plate,

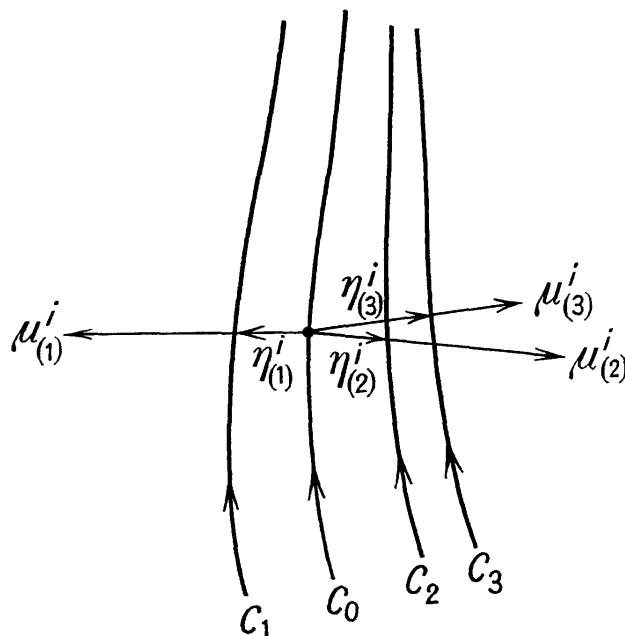


Fig. 7 – Frame of reference defined by C_0, C_1, C_2, C_3

or of the corner of a room. We consider also three timelike curves, C_1, C_2, C_3 , which can be controlled; actually two curves would do, but three are better for the sake of symmetry. These curves are to be thought of as the world lines of particles, adjustable by means of screws or other devices. The four world-lines *might* be four adjacent corners of a block of stone, but, if so, we must be prepared to strain that stone if necessary in order to satisfy the conditions which we shall now proceed to impose.

Each of the three adjustable world-lines has three degrees of freedom, and so we have at our disposal *nine* degrees of freedom in all. Let us so control them that the Born condition of rigidity is satisfied by each pair. Since there are six pairs, in this way we use up six degrees of freedom, leaving *three* over. Note that the controlling operations are physical (chronometric).

Without using up more degrees of freedom, we can make the control a little more definite by demanding that the three deviation vectors, drawn from C_0 , are equal in magnitude and mutually orthogonal. Then in the 3-element orthogonal to C_0 we have a little rigid tetra-

hedron $C_0C_1C_2C_3$ as in Fig. 8, which shows also the unit vectors $\mu_{(\alpha)}^i$ drawn in the directions of the deviation vectors $\eta_{(\alpha)}$. This orthonormal tetrad $\mu_{(\alpha)}^i$ forms a *frame of reference*.

In fact, *any* orthonormal triad, orthogonal to the 4-velocity of the observer's world-line C_0 , may be taken as a frame of reference. All we have been doing above is merely the conversion of this mathematical concept into physical (chronometric) terms, so that any astronomer anxious to set up a frame of reference will know how to go about it. Of all frames of reference, the simplest mathematically is given by a *Fermi triad* [cf. I-§ 4, II-§ 10], which, in the notation of II-(180), satisfies the equation

$$\frac{\delta}{\delta s} \lambda_{(\alpha)}^i = b A^i \lambda_{(\alpha)}^j B_j. \quad (27)$$

If $\mu_{(\alpha)}^i$ is any other orthonormal triad orthogonal to C_0 , there exists an orthogonal matrix $M_{(\alpha\beta)}$ (consisting of the nine mutual direction cosines) such that

$$\mu_{(\alpha)}^i = M_{(\alpha\beta)} \lambda_{(\beta)}^i, \quad M_{(\alpha\beta)} M_{(\alpha\gamma)} = \delta_{\beta\gamma}. \quad (28)$$

The elements $M_{(\alpha\beta)}$ are functions of s (time on C_0), and, since the derivative of an orthogonal matrix is skew-symmetric, we may write

$$\frac{d}{ds} M_{(\alpha\beta)} = \omega_{(\alpha\beta)} = -\omega_{(\beta\alpha)}, \quad (29)$$

the three independent components of $\omega_{(\alpha\beta)}$ being in fact the components of the *angular velocity* of $\mu_{(\alpha)}^i$ relative to $\lambda_{(\alpha)}^i$.

We have still at our disposal three degrees of freedom in the control of the world-lines C_1, C_2, C_3 which define $\mu_{(\alpha)}^i$, and we may use them to make $\omega_{(\alpha\beta)} = 0$. The frame then becomes a Fermi frame, and, as we shall see later, a Fermi frame is very useful physically.

What we have done above is partly physical and partly mathematical. The orthonormality of the frame $\mu_{(\alpha)}^i$ is physical (chronometric), but the Fermi law (27) is still merely mathematical. It is only

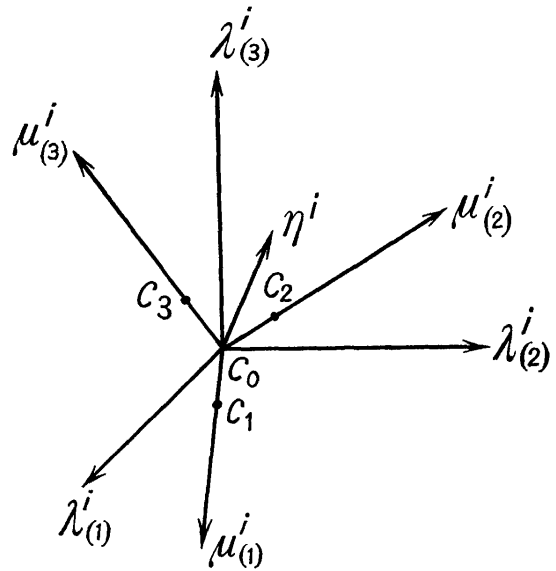


Fig. 8 – The frame of reference $\mu_{(\alpha)}^i$ viewed in the 3-element orthogonal to C_0 , and the Fermi triad $\lambda_{(\alpha)}^i$

after we have given a physical test of Fermi transport (as we shall later) that we can complete the physical specification. Meanwhile we shall use Fermi transport as if it were a physically understood thing.

Once we are equipped with a frame of reference $\mu_{(\alpha)}^i$ (Fermi or not), it is easy to write down a suitable definition of 3-momentum of a material particle or photon relative to that frame:

$$p_{(\alpha)} = p_i \mu_{(\alpha)}^i. \quad (30)$$

If we denote the 4-velocity of C_0 by $\mu_{(4)}^i$, we can combine 3-momentum and energy [cf. (23)] in the formulae

$$p_{(\alpha)} = p_i \mu_{(\alpha)}^i, \quad E = -p_{(4)} = p^{(4)}, \quad (31)$$

the rules of 1-(54) being understood.

§ 6. THE MEASUREMENT OF DIRECTION

The related concepts of *direction* and *angle* are so basic in Newtonian physics that it takes an effort to realize that they must be examined critically before they are admitted into relativity. It is a question of defining the direction of some object with world-line C relative to an observer with world-line C_0 , and we have to remember that C_0 can know nothing of C except through messages or signals of some sort which emanate from C and reach C_0 . These messages might be free particles or photons in vacuo, but we shall be more general and merely suppose that a message travels along some timelike or null world-line. In order that C_0 may be able to report the direction from which the message has come, he must set a trap for it by arranging to have another world-line C' adjacent to C_0 , adjusting C' so that the world-line of the message intersects C' . This is what an astronomer does in observing a star, C_0 being the world-line of his eye and C' that of the middle point of the object glass of his telescope.

If the distance between C_0 and C' is finite, various complications arise with which we do not wish to deal, because the observing apparatus is in practice very small compared with the phenomena observed; it will suffice to suppose the distance between C_0 and C' to be infinitesimal. Fig. 9 shows the message leaving the object C at the event P , cutting the world-line C' at the event P' , and finally reaching the observer C_0 at P_0 . It is the infinitesimal vector ξ^i , drawn from P_0 to P' , which we must use to define the direction of C relative to C_0 .

Let $\mu_{(\alpha)}^i$ be a frame of reference on C_0 , as in the preceding section.

Since ξ^i is not orthogonal to C_0 , it cannot be drawn in Fig. 8. However, if we proceed along C' to its intersection Q' with the 3-element orthogonal to C_0 at P_0 , we get a vector $\eta^i (= P_0Q')$ which can be drawn in Fig. 8, and which has direction ratios $\eta_i\mu_{(\alpha)}^i$ relative to the triad $\mu_{(\alpha)}^i$, taken at P_0 . We define the *direction of C relative to C_0* by these direction ratios, which are obviously equal to the components $\xi_i\mu_{(\alpha)}^i$. Naturally, these direction ratios depend not only on the frame used, but also on

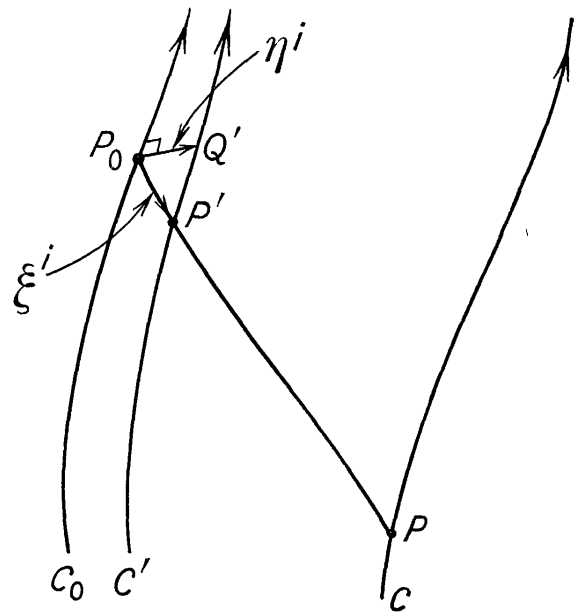


Fig. 9 – Observation of direction

the type of message employed. If, in astronomical observation, we neglect the refraction of the atmosphere, then PP_0 will be a null geodesic.

Since we are working with infinitesimals, it is a matter of indifference whether we take $\mu_{(\alpha)}^i$ at P_0 and η^i as shown, or $\mu_{(\alpha)}^i$ at the foot of the perpendicular from P' on C_0 and η^i drawn along this perpendicular. Such distinctions would become of importance only if we had to consider the length of the telescope or other observing apparatus as finite, i.e. if the time taken by the message from P' to P_0 were not negligible.

Direction might also be measured mechanically by using 3-momentum with the formula (30).

§ 7. RELATIVE VELOCITY AND THE DOPPLER EFFECT

It is a good thing to attach physical (chronometric) meanings to the mathematical formulae of Riemannian space-time. But it would be tedious to do this consistently, and it is more rewarding to go ahead with mathematical constructions (e.g. parallel transport) with a general confidence that they can be physically interpreted later if need be.

The 4-velocity of a particle is well defined; it is the unit tangent vector $V^i (= dx^i/ds)$ to the world-line of the particle. We now seek a useful definition of the velocity of one particle relative to another. Let C (Fig. 10) be the world-line of an observer and C' the world-line of

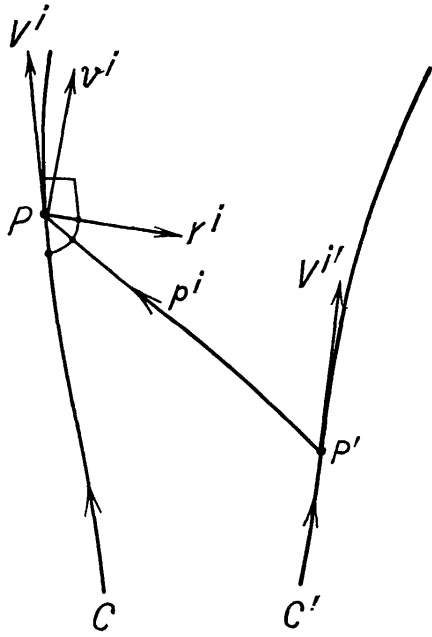


Fig. 10—Relative velocity and Doppler effect (mechanical)

some luminous object, such as a star or planet. We connect them with null geodesics such as $P'P$. We cannot immediately compare the 4-velocity V^i of C at P with the 4-velocity $V^{i'}$ of C' at P' , because they are vectors at different events. The obvious plan is to bring them to a common event by subjecting $V^{i'}$ to parallel transport along $P'P$; this gives us at P the vector

$$v_i = g_{ij'} V^{j'}, \quad (32)$$

where $g_{ij'}$ is the parallel propagator [II-(71)]. Let $\lambda_{(\alpha)}^i$ be a frame of reference on C with $\lambda_{(4)}^i = V^i$. This might be a Fermi frame, but the question does not arise at the moment, because we are concerned only with the event P .

We now define the *3-velocity of C' relative to C* by the three invariant components

$$v_{(\alpha)} = v_i \lambda_{(\alpha)}^i. \quad (33)$$

Since v^i and V^i are unit vectors, the fourth component

$$v_{(4)} = v_i \lambda_{(4)}^i = v_i V^i \quad (34)$$

is expressible in terms of the other three:

$$v^{(4)} = -v_{(4)} = (1 + v^2)^{\frac{1}{2}}, \quad v^2 = v_{(\alpha)} v^{(\alpha)}. \quad (35)$$

We may call v the *relative speed*. Note that $v^{(4)} = 1$ if, and only if, all the three components $v_{(\alpha)}$ vanish; in that case $V^{i'}$ and V^i are parallel for transport along the null geodesic $P'P$, and we may say that C' is *at rest* relative to C .

Let r^i be a unit vector at P , as shown in Fig. 10, orthogonal to C ($r_i V^i = 0$) and lying in the 2-element which contains the tangent at P to C and $P'P$. We define the *speed of recession* of C' to be

$$v_R = v_i r^i = v_{(\alpha)} r^{(\alpha)}, \quad (36)$$

where $r^{(\alpha)}$ are the components of r^i on the frame of reference.

We shall now discuss the Doppler effect in two ways: mechanically, and in terms of frequency.

In the mechanical treatment, we consider a photon having $P'P$ for

world-line, being emitted from C' at P' with 4-momentum $p^{i'}$, so that the energy of emission is, by (23),

$$E' = -p_{i'}V^{i'}. \quad (37)$$

The 4-momentum undergoes parallel transport along $P'P$ (cf. § 3), and so on arrival at P we have

$$p_i v^i = p_{i'} V^{i'}, \quad (38)$$

since a scalar product is constant under parallel transport of both the vectors in it. Thus, by (37),

$$p_{(\alpha)} v^{(\alpha)} + p_{(4)} v^{(4)} = -E'. \quad (39)$$

The energy of the photon relative to C is

$$E = -p_i V^i = -p_{(4)} = p^{(4)}, \quad (40)$$

and so (39) gives

$$E v^{(4)} - p_{(\alpha)} v^{(\alpha)} = E'. \quad (41)$$

From the definition of r^i and the fact p^i is null, we have

$$p^i = \theta(V^i - r^i), \quad (42)$$

where θ is some scalar, and, on multiplying by V_i we find

$$\theta = -p_i V^i = E. \quad (43)$$

Hence

$$p_{(\alpha)} v^{(\alpha)} = p_i \lambda^i_{(\alpha)} v^{(\alpha)} = -E r_{(\alpha)} v^{(\alpha)} = -E v_R, \quad (44)$$

and when we substitute this in (41) and use (35), we get the following relation between the *energy of reception* (E) and the *energy of emission* (E'):

$$E[(1 + v^2)^{\frac{1}{2}} + v_R] = E'. \quad (45)$$

This is the *Doppler effect in terms of energy*. The *red-shift* is given by

$$\frac{E' - E}{E'} = 1 - \frac{1}{(1 + v^2)^{\frac{1}{2}} + v_R}, \quad (46)$$

a negative red-shift being a violet-shift. If the relative speed is small, we get

$$\frac{E' - E}{E'} = v_R - v_R^2 + \frac{1}{2}v^2 + \dots, \quad (47)$$

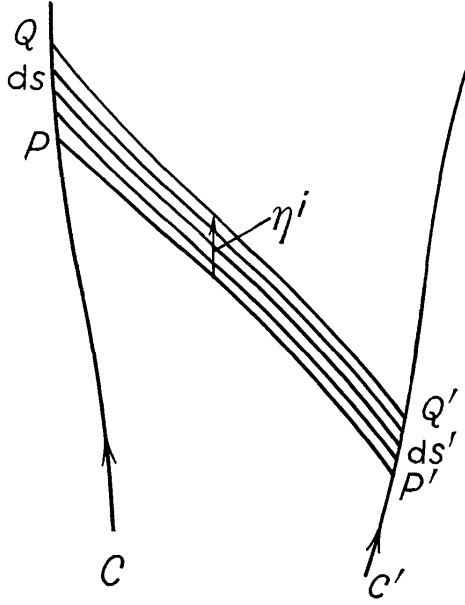


Fig. 11 – Doppler effect
(frequency)

in which the radial speed v_R is of course the dominant term.

To discuss the Doppler effect in terms of frequency, we consider (Fig. 11) a set of null geodesics joining C' to C , each representing the history of a wave crest. If there are n such crests, and ds' , ds are the clock-measures of $P'Q'$, PQ , respectively, we have

$$n = \nu' ds' = \nu ds, \quad (48)$$

where ν' , ν are the frequencies of emission and reception respectively. Thus the Doppler effect in terms of frequency is given by

$$\frac{\nu}{\nu'} = \frac{ds'}{ds}, \quad (49)$$

so that it involves only a comparison of the measures of corresponding elements when C' is mapped on C by null geodesics. This mapping was considered in I-§ 6, but with a different notation since V^i is now 4-velocity. If η^i is the deviation vector from $P'P$ to $Q'Q$, the second equation in I-(133) gives

$$\eta_i p^i = \eta'_i p'^i, \quad (50)$$

where p^i is the 4-momentum as earlier. But

$$\begin{aligned} \eta_i &= V_i ds, & \eta'_i &= V'_i ds', \\ \eta_i p^i &= -E ds, & \eta'_i p'^i &= -E' ds', \end{aligned} \quad (51)$$

where E and E' are the energies as earlier. Combining this with (49) and (50), we get $\nu/\nu' = ds'/ds = E/E'$, and so by (46) we have for the red-shift in terms of frequency

$$\frac{\nu' - \nu}{\nu'} = 1 - \frac{1}{(1 + v^2)^{\frac{1}{2}} + v_R} = v_R - v_R^2 + \frac{1}{2}v^2 + \dots \quad (52)$$

The quantum equation $E = h\nu$ (h = Planck's constant) has not been used in the above work; it is of course consistent with it.

It is clear that in general the observer of a luminous source will see a spectral shift — radiation emitted with frequency ν' relative to the source will be received with a different frequency ν relative to the

observer. In attributing a *cause* to this spectral shift, one would say, on inspecting (52), that the spectral shift was caused by the relative velocity of source and observer; it is in fact a Doppler effect in the original sense of the term. It is not a gravitational effect, because the Riemann tensor appears nowhere in our formulae ¹.

This statement contradicts a statement frequently made in general relativity to the effect that a gravitational field causes a red-shift. Arguments about this are completely futile because they are merely windy warfare conducted without any attempt to analyze the meanings of the terms employed. We have committed ourselves in (33) to a certain definition of relative velocity; if that definition is not accepted, then the statement attributing spectral shifts to relative velocity cannot be accepted either. Any confusion which may exist here is due to the excessive attention paid to statical gravitational fields, in which there is available a definition of velocity which has no meaning in non-static cases.

For later reference (cf. VII-§ 9, VIII-§ 3, XI-§ 9) we note that, by (37) and (40), the red-shift may be expressed in terms of the derivatives of the world-function $\Omega(P'P)$ as follows:

$$\frac{\nu' - \nu}{\nu'} = \frac{\Omega_{i'} V^{i'} + \Omega_i V^i}{\Omega_{j'} V^{j'}}. \quad (53)$$

§ 8. FERMI TRANSPORT AND THE BOUNCING PHOTON

We shall now describe an experiment by means of which an observer can find out whether some given frame of reference undergoes Fermi transport or not. The experiment is very simple, and we can describe it in ordinary physical terms. The observer might be a man on the earth's surface. His apparatus is a photon-gun, which in the present connection means a tube through which a photon can be shot out in the direction in which the gun is pointed and through which a photon can be received, as in a telescope. The observer launches some sort of small balloon into the air, and, taking careful aim, shoots from his gun a burst of photons to strike the balloon. These photons are scattered, and one comes back to the observer. If he is careless, he will not catch it, because his gun will probably be pointing in the wrong direction. But let us suppose that he catches it in the gun. Relative to

¹ This was pointed out by C. LANCZOS [1923c], who used a different mathematical argument.

any frame of reference which the observer may employ, there are then two directions of interest: the direction in which the photons are emitted from the gun, and the direction of the gun when the returning photon is received.

Now if the distance of the balloon is small of the first order, the trip-time of the photon is small of that same order, for, from the way in which we have defined length or distance, the 'velocity of light' is unity. If there are two frames of reference which have, relative to one another, a finite angular velocity, then in this small time one will turn relative to the other through a small angle of the first order. If the two directions of the gun (for emission and reception) happen to be the same relative to one frame, then they will differ by the first order when measured in the second frame.

If observations are made on a number of balloons and if, for some law of transport of the frame of reference, the directions of the gun for emission and reception are the same for each balloon, it is clear that this test picks out some particular law of transport. This is in fact Fermi transport, as we shall see when we have made the calculations. But since we may hope to extract more information from this problem of the *bouncing photon*, we shall carry out the calculations to a higher degree of accuracy than is actually required in order to establish this important property of Fermi transport.

The above explanation has been given in quasi-Newtonian form in order to convince a reader who is suspicious of space-time diagrams that the test proposed is a real physical test, allowance being made

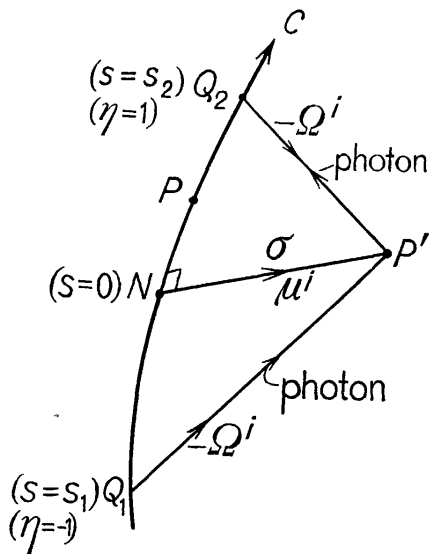


Fig. 12 – Fermi transport and the bouncing photon

for certain idealizations which would be made in any theoretical discussion. But indeed the matter is really much simpler when viewed in space-time, the whole story being told by Fig. 12, which shows the observer's world-line C , the event P' at which the photons strike the balloon and are scattered, the event of emission Q_1 and the event of reception Q_2 , Q_1P' and $P'Q_2$ being the null geodesics representing the histories of photons. To avoid encumbering the picture, the orthonormal triads representing the frame of reference at Q_1 and Q_2 are omitted.

As a mathematical construction, we draw the spacelike geodesic NP' orthogonal to C , and we mark at P a current event in the arc Q_1Q_2 . We denote by s the observer's time at P , with $s = 0$ at N . We write $NP' = \sigma$ and denote by μ^i the unit tangent to NP' at N . The approximations are controlled by the smallness of σ ($\sigma = O_1$).

Before starting the calculations, we note that the geometry of Fig.12 is determined in a given space-time by the curve C and the event P' . The curve C is defined by the event N and Cauchy data at N , viz. the 4-velocity A^i and its derivatives DA^i , D^2A^i ..., where $D = \delta/\delta s$, the absolute derivative operator. Then P' is determined by σ and μ^i . Thus, if $s = s_1$ at Q_1 and $s = s_2$ at Q_2 , s_1 and s_2 are determined by the quantities enumerated above. The first step in our calculations is to find s_1 and s_2 .

As we let P range along C , the world-function $\Omega(PP')$ is a function of s , and we may expand it in the following power series:

$$\begin{aligned}\Omega(PP') = & \Omega + s\Omega_i A^i + \frac{1}{2}s^2(\Omega_i DA^i + \Omega_{ij} A^i A^j) \\ & + \frac{1}{6}s^3(\Omega_i D^2 A^i + 3\Omega_{ij} DA^i \cdot A^j + \Omega_{ijk} A^i A^j A^k) \\ & + \frac{1}{24}s^4(\Omega_i D^3 A^i + 4\Omega_{ij} D^2 A^i \cdot A^j + 3\Omega_{ij} DA^i DA^j \\ & + 5\Omega_{ijk} DA^i \cdot A^j A^k + \Omega_{ijk} A^i A^j DA^k + \Omega_{ijkm} A^i A^j A^k A^m) + O_5 \quad (54)\end{aligned}$$

On the right $\Omega = \Omega(NP')$, and the differentiations are with respect to N . In working out the coefficients, we have used the fact that $\Omega_{ij} = \Omega_{ji}$ quite generally. Now

$$\Omega(NP') = \frac{1}{2}\sigma^2, \quad \Omega_i(NP') = -\sigma\mu_i, \quad (55)$$

accurately, and, expanding along NP' ,

$$\begin{aligned}\Omega_{ij}(NP') &= [\Omega_{ij}] + \sigma\mu^k[\Omega_{ijk}] + \frac{1}{2}\sigma^2\mu^k\mu^m[\Omega_{ijk'm'}] + O_3, \\ \Omega_{ijk}(NP') &= [\Omega_{ijk}] + \sigma\mu^m[\Omega_{ijkm}] + O_2, \\ \Omega_{ijkm}(NP') &= [\Omega_{ijkm}] + O_1,\end{aligned} \quad (56)$$

where $[]$ indicates a coincidence limit as $P' \rightarrow N$. Applying II-(69), we have

$$\begin{aligned}\Omega_{ij}(NP') &= g_{ij} + \frac{1}{2}\sigma^2 S_{ijkm}\mu^k\mu^m + O_3, \\ \Omega_{ijk}(NP') &= -\sigma S_{ijkm}\mu^m + O_2, \\ \Omega_{ijkm}(NP') &= S_{ijkm} + O_1,\end{aligned} \quad (57)$$

where g_{ij} and the symmetrized Riemann tensor S_{ijkm} are evaluated

at N . We now substitute from (55) and (57) in (54) and use the following equations

$$\begin{aligned} A_i \mu^i &= 0, & A_i A^i &= -1, & A_i D A^i &= 0, \\ A_i D^2 A^i + D A_i D A^i &= 0. \end{aligned} \quad (58)$$

We get

$$\begin{aligned} \Omega(PP') &= \frac{1}{2}\sigma^2 - \frac{1}{2}s^2(\sigma\mu_i D A^i + 1 - \frac{1}{2}\sigma^2 S_{ijkm} A^i A^j \mu^k \mu^m) \\ &\quad - \frac{1}{6}s^3 \sigma \mu_i D^2 A^i - \frac{1}{24}s^4 D A_i D A^i + O_5. \end{aligned} \quad (59)$$

Now by II-(69) and I-(99) we have

$$S_{ijkm} A^i A^j \mu^k \mu^m = -\frac{2}{3} R_{ikjm} A^i \mu^k A^j \mu^m = \frac{2}{3} K, \quad (60)$$

where K is the Riemannian curvature of the 2-element defined at N by A^i and μ^i . We have also

$$D A_i D A^i = b^2, \quad (61)$$

where b is the first curvature of C at N . Thus we may rewrite (59) in the form

$$\begin{aligned} 2\Omega(PP') &= \sigma^2 - s^2(1 + \sigma\mu_i D A^i - \frac{1}{3}\sigma^2 K) \\ &\quad - \frac{1}{3}s^3 \sigma \mu_i D^2 A^i - \frac{1}{12}s^4 b^2 + O_5. \end{aligned} \quad (62)$$

To find s_1 and s_2 , we are to put $\Omega(PP') = 0$. It is clear that

$$s_1, s_2 = \eta\sigma + O_2, \quad (63)$$

where $\eta = -1$ for s_1 and $\eta = 1$ for s_2 . Then (62) gives for s_1, s_2 the simple quadratic equation

$$s^2(1 + \sigma\mu_i D A^i - \frac{1}{3}\sigma^2 K) = M(\sigma, \eta) + O_5 \quad (64)$$

where

$$M(\sigma, \eta) = \sigma^2 - \frac{1}{3}\eta\sigma^4 \mu_i D^2 A^i - \frac{1}{12}\sigma^4 b^2. \quad (65)$$

Hence we find

$$s_1 = -\frac{1}{2}\tau + \frac{1}{2}f + O_4, \quad s_2 = \frac{1}{2}\tau + \frac{1}{2}f + O_4, \quad (66)$$

where

$$\begin{aligned} \tau &= 2\sigma - \sigma^2 \mu_i D A^i + \sigma^3(\frac{1}{3}K - \frac{1}{12}b^2 + \frac{3}{4}(\mu_i D A^i)^2), \\ f &= -\frac{1}{3}\sigma^3 \mu_i D^2 A^i. \end{aligned} \quad (67)$$

Thus the trip-time from emission at Q_1 to return at Q_2 is

$$s_2 - s_1 = \tau + O_4. \quad (68)$$

We note that if C is a geodesic, then

$$\tau = 2\sigma + \frac{1}{3}K\sigma^3. \quad (69)$$

So far the directions of the emitted photon and the returning photon have not appeared. To discuss these directions, we introduce a Fermi tetrad $\lambda_{(a)}^i$ on C ; each of the four vectors satisfies the equation of F-W transport I-(72)

$$D\lambda^i = \lambda_j(A^i D A^j - A^j D A^i), \quad (70)$$

and $\lambda_{(4)}^i = A^i$, the current 4-velocity of C . Now $-\Omega^i(Q_1 P')$, $-\Omega^i(Q_2 P')$ are null vectors at Q_1 , Q_2 tangent to the null geodesics $Q_1 P'$, $Q_2 P'$, pointing from the observer towards the balloon as indicated in Fig. 12. The direction of emission is of course that of the emitted photon, and the direction of reception is that of the returning photon reversed. Thus the direction cosines of the directions of emission and reception are of the form

$$\theta_{(\alpha)} = -\chi \Omega_i \lambda_{(\alpha)}^i = -\chi \Omega_{(\alpha)}, \quad (71)$$

where χ is a *positive* scalar such that

$$\chi^{-2} = \Omega_{(\alpha)} \Omega^{(\alpha)} = -\Omega_{(4)} \Omega^{(4)} = (\Omega_{(4)})^2, \quad (72)$$

from the null character of Ω^i . Now

$$\begin{aligned} \Omega^{(4)} &< 0, & \Omega_{(4)} &> 0 \text{ at } Q_1, \\ \Omega^{(4)} &> 0, & \Omega_{(4)} &< 0 \text{ at } Q_2, \end{aligned}$$

and so, with η as after (63), we include the values at χ at the two events in the formula

$$\chi = -\frac{\eta}{\Omega_{(4)}}. \quad (73)$$

Thus the direction cosines for emission and reception are given by

$$\theta_{(\alpha)} = \eta \frac{\Omega_{(\alpha)}}{\Omega_{(4)}} = \eta \frac{\Omega_i \lambda_{(\alpha)}^i}{\Omega_j \lambda_{(4)}^j} = \eta \frac{\Omega_i \lambda_{(\alpha)}^i}{\Omega_j A^j}. \quad (74)$$

Our object is to calculate the changes in these direction cosines in passing from Q_1 to Q_2 .

The calculations are best effected by first dropping the labels on the vectors of the Fermi tetrad and indeed forgetting about (70), so that we have an arbitrary vector field λ^i on C . Then the quantity

$$\phi(PP') = \Omega_i(PP') \lambda^i(P) \quad (75)$$

is a function of s on C , and we may expand it as follows:

$$\begin{aligned}\phi(PP') &= \Omega_i \lambda^i + s(\Omega_i D\lambda^i + \Omega_{ij} \lambda^i A^j) \\ &+ \frac{1}{2}s^2(\Omega_i D^2\lambda^i + 2\Omega_{ij} D\lambda^i A^j + \Omega_{ij} \lambda^i DA^j + \Omega_{ijk} \lambda^i A^j A^k) \\ &+ \frac{1}{6}s^3(\Omega_i D^3\lambda^i + 3\Omega_{ij} D^2\lambda^i A^j + 3\Omega_{ij} D\lambda^i DA^j + \Omega_{ij} \lambda^i D^2A^j \\ &+ 2\Omega_{ijk} \lambda^i DA^j A^k + 3\Omega_{ijk} D\lambda^i A^j A^k + \Omega_{ijk} \lambda^i A^j DA^k \\ &+ \Omega_{ijk} \lambda^i A^j A^k A^m) + O_4.\end{aligned}\quad (76)$$

On the right hand side $\Omega = \Omega(NP')$, with derivatives with respect to N , and $\lambda^i = \lambda^i(N)$. We now substitute from (55) and (57); this gives

$$\begin{aligned}\phi(PP') &= -\sigma\mu_i \lambda^i + s(\lambda_i A^i - \sigma\mu_i D\lambda^i + \frac{1}{2}\sigma^2 S_{ijk} \lambda^i A^j \mu^k \mu^m) \\ &+ \frac{1}{2}s^2(2A_i D\lambda^i + \lambda_i DA^i - \sigma\{\mu_i D^2\lambda^i + S_{ijk} \lambda^i A^j A^k \mu^m\}) \\ &+ \frac{1}{6}s^3(3A_i D^2\lambda^i + 3DA_i D\lambda^i + \lambda_i D^2A^i) + O_4.\end{aligned}\quad (77)$$

We may simplify this slightly by using an implication of the F-W law (70); we have

$$\begin{aligned}A_i \lambda^i &= \text{const.}, \\ A_i D\lambda^i + DA_i \lambda^i &= 0, \\ A_i D^2\lambda^i + 2DA_i D\lambda^i + D^2A_i \lambda^i &= 0, \\ 2A_i D\lambda^i + DA_i \lambda^i &= -DA_i \lambda^i, \\ 3A_i D^2\lambda^i + 3DA_i D\lambda^i + \lambda_i D^2A^i &= -2D^2A_i \lambda^i - 3DA_i D\lambda^i.\end{aligned}\quad (78)$$

The last two lines may be substituted in (77).

The coefficients of the powers of s and σ in (77) are evaluated at N . To find $\phi(Q_1P')$ and $\phi(Q_2P')$, we are to substitute, as in (66),

$$\begin{aligned}s &= \frac{1}{2}\eta\tau + \frac{1}{2}f + O_4, & f &= O_3, \\ s^2 &= \frac{1}{4}\tau^2 + O_4, & s^3 &= \frac{1}{8}\eta\tau^3 + O_4,\end{aligned}\quad (79)$$

with $\eta = -1$ at Q_1 and $\eta = 1$ at Q_2 . Hence, with Q standing for Q_1 or Q_2 ,

$$\begin{aligned}\phi(QP') &= -\sigma\mu_i \lambda^i + \frac{1}{2}\eta\tau(A_i \lambda^i - \sigma\mu_i D\lambda^i + \frac{1}{2}\sigma^2 S_{ijk} \lambda^i A^j \mu^k \mu^m) \\ &+ \frac{1}{2}fA_i \lambda^i - \frac{1}{8}\tau^2(DA_i \lambda^i + \sigma\{\mu_i D^2\lambda^i + S_{ijk} \lambda^i A^j A^k \mu^m\}) \\ &- \frac{1}{48}\eta\tau^3(2D^2A_i \lambda^i + 3DA_i D\lambda^i) + O_4.\end{aligned}\quad (80)$$

We have now to substitute from (67)

$$\begin{aligned}\frac{1}{2}\tau &= \sigma - \frac{1}{2}\sigma^2\mu_i DA^i + \sigma^3E, & E &= \frac{1}{6}K - \frac{1}{24}b^2 + \frac{3}{8}(\mu_i DA^i)^2, \\ \frac{1}{2}f &= -\frac{1}{6}\sigma^3\mu_i D^2A^i.\end{aligned}\quad (81)$$

We obtain

$$\phi(QP') = (W + \eta W^*)\sigma + (Y + \eta Y^*)\sigma^2 + (Z + \eta Z^*)\sigma^3 + O_4, \quad (82)$$

where the coefficients have the following values:

$$\begin{aligned} W &= -\mu_i \lambda^i, & W^* &= A_i \lambda^i, \\ Y &= -\frac{1}{2} D A_i \lambda^i, & Y^* &= -\mu_i D \lambda^i - \frac{1}{2} \mu_i D A^i A_j \lambda^j, \\ Z &= \frac{1}{2} \mu_i D A^i D A_j \lambda^j - \frac{1}{2} \mu_i D^2 \lambda^i - \frac{1}{6} \mu_i D^2 A^i A_j \lambda^j - \frac{1}{2} S_{ijkm} \lambda^i A^j A^k \mu^m, \\ Z^* &= E A_i \lambda^i + \frac{1}{2} \mu_i D A^i \mu_j D \lambda^j - \frac{1}{3} D^2 A_i \lambda^i - \frac{1}{2} D A_i D \lambda^i \\ &\quad + \frac{1}{2} S_{ijkm} \lambda^i A^j \mu^k \mu^m. \end{aligned} \quad (83)$$

We now restore to λ^i the labels indicating a Fermi tetrad. If we write $\lambda_{(\alpha)}^i$ for λ^i , $\phi(QP')$ becomes $\Omega_{(\alpha)}(QP')$, and if we write $A^i (= \lambda_{(4)}^i)$ for λ^i , $\phi(QP')$ becomes $\Omega_{(4)}(QP')$. These are the quantities we need in (74) for the calculation of the direction cosines, and we proceed to evaluate them.

By the Frenet-Serret formulae 1-(55) we have

$$\begin{aligned} D A^i &= b B^i, \\ D B^i &= c C^i + b A^i, \\ D^2 A^i &= b^2 A^i + D b B^i + b c C^i. \end{aligned} \quad (84)$$

Denoting components on the Fermi tetrad $\lambda_{(\alpha)}^i$ in the usual way, we have

$$\begin{aligned} D \lambda_{(\alpha)}^i &= A^i \lambda_{(\alpha)}^j D A_j = b A^i \lambda_{(\alpha)}^j B_j = b A^i B_{(\alpha)}, \\ D^2 \lambda_{(\alpha)}^i &= A^i (D b B_{(\alpha)} + b c C_{(\alpha)}) + B^i b^2 B_{(\alpha)}. \end{aligned} \quad (85)$$

Then

$$\begin{aligned} \mu_i A^i &= 0, & \mu_i \lambda_{(\alpha)}^i &= \mu_{(\alpha)}, \\ \mu_i D A^i &= b B_{(\alpha)} \mu^{(\alpha)}, & \mu_i D \lambda_{(\alpha)}^i &= 0, \\ \mu_i D^2 A^i &= D b B_{(\alpha)} \mu^{(\alpha)} + b c C_{(\alpha)} \mu^{(\alpha)}, & \mu_i D^2 \lambda_{(\alpha)}^i &= b^2 B_{(\alpha)} B_{(\beta)} \mu^{(\beta)}, \end{aligned} \quad (86)$$

and

$$\begin{aligned} A_i \lambda_{(\alpha)}^i &= 0, & D A_i \lambda_{(\alpha)}^i &= b B_{(\alpha)}, & D A_i D \lambda_{(\alpha)}^i &= 0, \\ D^2 A_i \lambda_{(\alpha)}^i &= D b B_{(\alpha)} + b c C_{(\alpha)}, \end{aligned} \quad (87)$$

and also

$$\begin{aligned} S_{ijkm} \lambda_{(\alpha)}^i A^j \mu^k \mu^m &= S_{(\alpha 4 \beta \gamma)} \mu^{(\beta)} \mu^{(\gamma)}, \\ S_{ijkm} \lambda_{(\alpha)}^i A^j A^k \mu^m &= S_{(\alpha 4 4 \beta)} \mu^{(\beta)}. \end{aligned} \quad (88)$$

Note that $\mu_{(\alpha)}$ ($= \mu^{(\alpha)}$) are the direction cosines of NP' relative to the Fermi triad.

Inserting the label (α) in (83) and using the above results, we find

$$\begin{aligned} W_{(\alpha)} &= -\mu_{(\alpha)}, & W_{(\alpha)}^* &= 0, \\ Y_{(\alpha)} &= -\frac{1}{2}bB_{(\alpha)}, & Y_{(\alpha)}^* &= 0, \\ Z_{(\alpha)} &= -\frac{1}{2}S_{(\alpha 44\beta)}\mu^{(\beta)}, \\ Z_{(\alpha)}^* &= -\frac{1}{3}(DbB_{(\alpha)} + bcC_{(\alpha)}) + \frac{1}{2}S_{(\alpha 4\beta\gamma)}\mu^{(\beta)}\mu^{(\gamma)}. \end{aligned} \quad (89)$$

Thus (82) gives

$$\Omega_{(\alpha)}(QP') = W_{(\alpha)}\sigma + Y_{(\alpha)}\sigma^2 + (Z_{(\alpha)} + \eta Z_{(\alpha)}^*)\sigma^3 + O_4, \quad (90)$$

with the above values for the coefficients.

We now write A^i ($= \lambda_{(4)}^i$) for λ^i in (83) and obtain

$$\begin{aligned} W_{(4)} &= 0, & W_{(4)}^* &= -1, \\ Y_{(4)} &= 0, & Y_{(4)}^* &= -\frac{1}{2}bB_{(\alpha)}\mu^{(\alpha)}, \\ Z_{(4)} &= -\frac{1}{3}\mu_i D^2 A^i = -\frac{1}{3}(DbB_{(\alpha)}\mu^{(\alpha)} + bcC_{(\alpha)}\mu^{(\alpha)}), \\ Z_{(4)}^* &= -E + \frac{1}{2}b^2(B_{(\alpha)}\mu^{(\alpha)})^2 - \frac{1}{6}b^2 + \frac{1}{2}S_{(44\beta\gamma)}\mu^{(\beta)}\mu^{(\gamma)}. \end{aligned} \quad (91)$$

We can reduce this last expression, for, by (60) and (81),

$$\begin{aligned} K &= \frac{3}{2}S_{ijkm}A^i A^j \mu^k \mu^m = \frac{3}{2}S_{(44\beta\gamma)}\mu^{(\beta)}\mu^{(\gamma)}, \\ E &= \frac{1}{6}K - \frac{1}{24}b^2 + \frac{3}{8}b^2(B_{(\alpha)}\mu^{(\alpha)})^2, \end{aligned} \quad (92)$$

and so

$$Z_{(4)}^* = \frac{1}{6}K - \frac{1}{8}b^2 + \frac{1}{8}b^2(B_{(\alpha)}\mu^{(\alpha)})^2 = \frac{1}{6}K - \frac{1}{8}b^2 \sin^2(B\mu), \quad (93)$$

where $(B\mu)$ indicates the angle between the first normal B^i and the vector μ^i . We have then from (82)

$$\Omega_{(4)}(QP') = -\eta\sigma + \eta Y_{(4)}^*\sigma^2 + (Z_{(4)} + \eta Z_{(4)}^*)\sigma^3 + O_4. \quad (94)$$

Accordingly, by (74), (90) and (94), the direction cosines for emission ($\eta = -1$) and reception ($\eta = 1$) have the values

$$\theta_{(\alpha)} = \eta \frac{\Omega_{(\alpha)}}{\Omega_{(4)}} = \frac{W_{(\alpha)} + Y_{(\alpha)}\sigma + (Z_{(\alpha)} + \eta Z_{(\alpha)}^*)\sigma^2 + O_3}{-1 + Y_{(4)}^*\sigma + (Z_{(4)}^* + \eta Z_{(4)})\sigma^2 + O_3}. \quad (95)$$

By binomial expansion of the denominator, this becomes

$$\theta_{(\alpha)} = \psi_{(\alpha)} + \eta\psi_{(\alpha)}^* + O_3 \quad (96)$$

where

$$\begin{aligned} \psi_{(\alpha)} = & -W_{(\alpha)} - (Y_{(\alpha)} + W_{(\alpha)}Y_{(4)}^*)\sigma \\ & - \{Y_{(\alpha)}Y_{(4)}^* + Z_{(\alpha)} + W_{(\alpha)}(Z_{(4)}^* + Y_{(4)}^{*2})\}\sigma^2, \end{aligned} \quad (97)$$

$$\psi_{(\alpha)}^* = -\sigma^2(Z_{(\alpha)}^* + W_{(\alpha)}Z_{(4)}).$$

The increments in the direction cosines are therefore

$$\Delta\theta_{(\alpha)} = 2\psi_{(\alpha)}^* + O_3. \quad (98)$$

This equation tells how the photon gun must be turned, relative to the Fermi frame, in order that it may catch the returning photon. The rotation is of the second order in σ . *This fact serves to identify Fermi transport physically.* For if we use a frame which rotates relative to the Fermi frame, then in the time of the experiment (approximately 2σ), there will be a rotation of order σ relative to the Fermi frame; an observer who uses the rotating frame will have to turn his photon-gun through an angle of order σ in order to catch the returning photon. *Thus Fermi transport is distinguished by the fact that for Fermi frames, and for them alone, the angle of rotation of the photon-gun is zero if quantities of order σ^2 are neglected.* Or we may put it this way: if τ is the trip-time and θ the angle through which the photon gun must be turned, then the limit of θ/τ , as τ tends to zero, is zero for Fermi frames and for them alone.

Substitution in (97) from (89) and (91) gives

$$\begin{aligned} \psi_{(\alpha)} = & \mu_{(\alpha)}\{1 - \tfrac{1}{2}b\sigma \cos(B\mu) + \tfrac{1}{6}K\sigma^2 - \tfrac{1}{8}b^2\sigma^2(1 - 3\cos^2(B\mu))\} \\ & + B_{(\alpha)}(\tfrac{1}{2}b\sigma - \tfrac{1}{4}b^2\sigma^2 \cos(B\mu)) + \tfrac{1}{2}\sigma^2 S_{(\alpha 44\beta)}\mu^{(\beta)}, \end{aligned} \quad (99)$$

$$\psi_{(\alpha)}^* = \sigma^2(F_{(\alpha)} - \mu_{(\alpha)}F_{(\beta)}\mu^{(\beta)} - \tfrac{1}{2}S_{(\alpha 4\beta\gamma)}\mu^{(\beta)}\mu^{(\gamma)}),$$

where

$$F_{(\alpha)} = \tfrac{1}{3}(DbB_{(\alpha)} + bcC_{(\alpha)}). \quad (100)$$

This displays these 3-vectors in terms of components along $\mu_{(\alpha)}$ (the Fermi direction of the balloon), $B_{(\alpha)}$ and $C_{(\alpha)}$ (the first and second normals to the observer's world-line), and two other 3-vectors involving the symmetrized Riemann tensor II-(48).

As a check on these rather involved calculations, it is easy to verify

that $\theta_{(\alpha)}\theta^{(\alpha)} = 1 + O_3$, and that $\theta_{(\alpha)}$ and $\Delta\theta_{(\alpha)}$ are orthogonal to the order considered.

Since in a first approximation we have $\theta_{(\alpha)} = \mu_{(\alpha)}$ and $\tau = 2\sigma$ (τ being the trip-time), we can write (98) in the form

$$2\tau^{-2}\Delta\theta_{(\alpha)} = F_{(\alpha)} - \theta_{(\alpha)}F_{(\beta)}\theta^{(\beta)} - \frac{1}{2}S_{(\alpha\beta\gamma)}\theta^{(\beta)}\theta^{(\gamma)}. \quad (101)$$

In this equation all the quantities are observable except the F -terms and the S -terms. Thus, with sufficiently refined apparatus, observation of the bouncing photon would yield information about the curvatures of the observer's world-line and about the Riemann tensor.

We must however bear in mind that the above approximations are based on power series expansions, with σ small. This may be adequate for the discussion of a photon bouncing off an artificial satellite, but it certainly would not do for a photon bouncing off the moon.

Actually a process of approximation based solely on the smallness of σ is not admissible at all, since σ is a dimensional quantity (time or, equivalently, distance). Only dimensionless invariants can be called *small* in any absolute sense, and what really matters in the approximation is the ratio of a rejected term to those which are retained. However, the correct procedure (with precise statement of the assumptions made) would be somewhat tedious, and the interested reader should find little difficulty in filling in the necessary details in the above argument and in other similar arguments involving similar approximations ¹.

§ 9. THE FALLING APPLE

According to the famous legend, Newton was inspired to create his theory of gravitation by witnessing the fall of an apple from the branch of a tree, and students of Newtonian physics even today would say that the acceleration (980 cm sec^{-2}) of a falling apple is due to gravity. According to the theory of relativity, that view is quite wrong. We shall make a careful study of the problem, and we shall find that the gravitational field (i.e. the Riemann tensor) plays a very small part indeed in the phenomenon of free fall, the acceleration of 980 cm sec^{-2} being in fact due to the curvature of the world-line of the branch of the tree. Indeed, we would witness an acceleration of 980 cm sec^{-2} in an apple if we released it from a rocket travelling

¹ Cf. remarks on smallness in II-§ 3.

with that acceleration in remote space, far from any significant gravitational field. But it would only be confusing to pursue the matter further in Newtonian terms, or to refer to Einstein's principle of equivalence¹, because the problem is a problem in Riemannian geometry, simple in principle although a little complicated in detail. Merely for purposes of comparison and physical interpretation, we shall link the final results with some Newtonian ideas.

In Fig. 13, C is the world-line of an observer (the branch of the tree) and I' that of a freely falling body (the apple); I' is a geodesic, but C is not. C and I' touch at O , the event of release, the tangency corresponding to gentle release, without initial relative velocity. To follow the fall in a completely physical manner, we should consider signals passing from I' to C , but we shall be contented here with a more mathematical treatment².

Let P' be any event on I' and PP' the (spacelike) geodesic drawn orthogonal to C . Let $\sigma = PP'$ and let μ^i be the unit vector tangent to PP' at P . We shall study the vector $\sigma\mu^i$ by taking its component on a vector λ^i which is transported along C in any manner (we shall specialize the law of transport later).

Let $s = OP$, $s' = OP'$, these being times registered by clocks on the branch and on the apple respectively. The construction sets up a correspondence between s and s' , and we shall write

$$S = ds'/ds. \quad (102)$$

The invariant $\sigma\mu_i\lambda^i$ is a function of s . In terms of the world-function we have

$$\sigma\mu_i\lambda^i = -\Omega_i(PP')\lambda^i(P). \quad (103)$$

We shall expand this as a power series in s , but to avoid writing a lot

¹ Cf. MØLLER [1952, p. 220].

² For various reasons the calculations are carried out to a high order of approximation and that makes them look formidable. To get a quick rough view of the falling apple, the series may be cut off with rejection of the terms in s^3 ; then the right hand side of (125) will reduce to its first term, giving the relativistic analogue of the elementary problem of free fall.

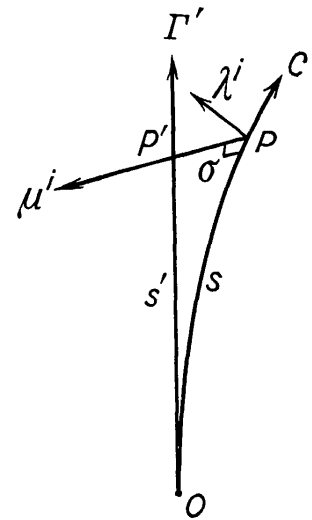


Fig. 13 – Free fall

of minus signs, we shall define

$$\phi(s) = \Omega_i(PP')\lambda^i(P), \quad (104)$$

and expand this in the series

$$\phi(s) = [\phi] + s[D\phi] + \frac{1}{2}s^2[D^2\phi] + \frac{1}{6}s^3[D^3\phi] + \frac{1}{24}s^4[D^4\phi] + O_5, \quad (105)$$

where $D = \delta/\delta s$ and $[\]$ means evaluation at O , i.e. a coincidence limit, since P' coincides with P when $s = 0$.

Let $A^i (= dx^i/ds)$ be the 4-velocity of C at P , and let $A^{i'}$ be the 4-velocity of Γ' at P' ; since Γ' is a geodesic, we have

$$DA^{i'} = 0. \quad (106)$$

We note that

$$\Omega_i A^i = 0, \quad (107)$$

on account of the orthogonality at P .

The formulae which follow resemble (76), but are more complicated since P' is not fixed. We have

$$D\phi = \Omega_i D\lambda^i + \Omega_{ij}\lambda^i A^j + \Omega_{ij'}\lambda^i A^{j'} S, \quad (108)$$

$$\begin{aligned} D^2\phi = & \Omega_i D^2\lambda^i + \Omega_{ij}(2D\lambda^i A^j + \lambda^i DA^j) + \Omega_{ij'}(2D\lambda^i A^{j'} S + \lambda^i A^{j'} DS) \\ & + \Omega_{ijk}\lambda^i A^j A^k + 2\Omega_{ijk'}\lambda^i A^j A^{k'} S + \Omega_{ij'k'}\lambda^i A^{j'} A^{k'} S^2, \end{aligned} \quad (109)$$

$$\begin{aligned} D^3\phi = & \Omega_i D^3\lambda^i + \Omega_{ij}(3D^2\lambda^i A^j + 3D\lambda^i DA^j + \lambda^i D^2 A^j) \\ & + \Omega_{ij'}(3D^2\lambda^i A^{j'} S + 2D\lambda^i A^{j'} DS + \lambda^i A^{j'} D^2 S) \\ & + \Omega_{ijk}(3D\lambda^i A^j A^k + 2\lambda^i DA^j A^k + \lambda^i A^j DA^k) \\ & + \Omega_{ijk'}(6D\lambda^i A^j A^{k'} S + 3\lambda^i DA^j A^{k'} S + 3\lambda^i A^j A^{k'} DS) \\ & + \Omega_{ij'k'}(3D\lambda^i A^{j'} A^{k'} S^2 + 3\lambda^i A^{j'} A^{k'} SDS) \\ & + \Omega_{ijkm}\lambda^i A^j A^k A^m + 3\Omega_{ijkm'}\lambda^i A^j A^k A^{m'} S \\ & + 3\Omega_{ijk'm'}\lambda^i A^j A^{k'} A^{m'} S^2 + \Omega_{ij'k'm'}\lambda^i A^{j'} A^{k'} A^{m'} S^3. \end{aligned} \quad (110)$$

Now take $\lambda^i = A^i$. Then, by (107), $\phi = 0$; hence $D\phi$, $D^2\phi$, $D^3\phi$ all vanish, and so do their coincidence limits. Referring to the list of coincidence limits in II-(69), we obtain from (108)

$$[S] = 1, \quad (111)$$

as is indeed obvious otherwise. From (109) we get

$$[DS] = 0, \quad (112)$$

and from (110) we get

$$[D^2S] = -2b^2, \quad b^2 = DA_i DA^i, \quad (113)$$

b being the first curvature of C at O . In making the above calculations, we use the fact that $[A^{i'}] = [A^i]$ and the fact that $[\Omega_{ijkm}]$ vanishes when contracted with the same vector three times.

We now take λ^i orthogonal to C , so that

$$\lambda^i A_i = 0, \quad (114)$$

but without any other restriction for the present. Using (111)–(113), we get from (104) and (108)–(110) the following coincidence limits:

$$[\phi] = 0, \quad [D\phi] = 0, \quad (115)$$

$$[D^2\phi] = \lambda^i DA_i, \quad (116)$$

$$[D^3\phi] = 3D\lambda^i DA_i + \lambda^i D^2A_i, \quad (117)$$

the right hand sides being evaluated at O . We need $[D^4\phi]$ also. It is unnecessary to write out $D^4\phi$, for we can carry out the differentiation of (110) mentally and get the coincidence limit in one step, because many terms vanish. Thus we get

$$\begin{aligned} [D^4\phi] = 6D^2\lambda^i DA_i + 4D\lambda^i D^2A_i + \lambda^i D^3A_i + 6b^2 D\lambda^i A_i \\ + R_{ijkm} \lambda^i A^j A^k DA^m; \end{aligned} \quad (118)$$

we have used $S_{imjk} - S_{ijmk} = R_{ikjm}$ [cf. II–(69)]. Therefore, for any λ^i satisfying (114), the series (105) reads

$$\phi(s) = \frac{1}{2}s^2[D^2\phi] + \frac{1}{6}s^3[D^3\phi] + \frac{1}{24}s^4[D^4\phi] + O_5, \quad (119)$$

the coefficients being as in (116)–(118).

There are two particularly interesting choices of λ^i , consistent with (114). First, we may take it to be a member of a Fermi triad; secondly, we may take it to be one of the principal normals of C .

Taking λ^i to be a Fermi vector, we have the following equations:

$$\begin{aligned} D\lambda^i &= A^i \lambda^j DA_j, \\ D\lambda^i A_i &= -\lambda^i DA_i, \quad D\lambda^i DA_i = 0, \quad D\lambda^i D^2A_i = -b^2 \lambda^i DA_i, \\ D^2\lambda^i &= DA^i \lambda^j DA_j + A^i D\lambda^j DA_j + A^i \lambda^j D^2A_j, \\ D^2\lambda^i DA_i &= b^2 \lambda^i DA_i. \end{aligned} \quad (120)$$

Hence

$$\begin{aligned} \phi(s) = \frac{1}{2}s^2\lambda^i DA_i + \frac{1}{6}s^3\lambda^i D^2 A_i + \frac{1}{24}s^4(\lambda^i D^3 A_i - 4b^2\lambda^i DA_i \\ + R_{ijkm}\lambda^i A^j A^k DA^m) + O_5. \end{aligned} \quad (121)$$

We shall now insert the label (α) , $\lambda_{(\alpha)}^i$ being a Fermi triad. Then, by II-(185), the first three Fermi coordinates of P' are

$$X_{(\alpha)} = -\Omega_i \lambda_{(\alpha)}^i, \quad (122)$$

i.e. ϕ with sign reversed. Thus, *in terms of Fermi coordinates, the fall of the apple is given by*

$$\begin{aligned} X_{(\alpha)} = -\frac{1}{2}s^2(DA)_{(\alpha)} - \frac{1}{6}s^3(D^2 A)_{(\alpha)} \\ + \frac{1}{24}s^4(4b^2(DA)_{(\alpha)} - (D^3 A)_{(\alpha)} - R_{(\alpha 44\beta)}(DA)^{(\beta)}) + O_5. \end{aligned} \quad (123)$$

Here s is the time registered by a clock on the branch of the tree (it is in fact the fourth Fermi coordinate), and the labels indicate components on the Fermi triad, evaluated at O , the event of release.

We can write this formula in a different form by expressing the derivatives of the 4-velocity A^i in terms of the principal normals and curvatures of C . By the Frenet-Serret formulae I-(55), we have

$$\begin{aligned} DA^i &= bB^i, \\ D^2 A^i &= b^2 A^i + DbB^i + bcC^i, \\ D^3 A^i &= 3bDbA^i + (D^2 b + b^3 - bc^2)B^i \\ &\quad + (bDc + 2cDb)C^i + bcdD^i, \end{aligned} \quad (124)$$

where B^i , C^i , D^i are the first, second and third unit normals, and b , c , d the first, second and third curvatures of C . Then (123) may be written

$$\begin{aligned} X_{(\alpha)} = -\frac{1}{2}s^2 b B_{(\alpha)} - \frac{1}{6}s^3 (DbB_{(\alpha)} + bcC_{(\alpha)}) \\ + \frac{1}{24}s^4 \{3b^3 - D^2 b + bc^2\} B_{(\alpha)} - (bDc + 2cDb)C_{(\alpha)} \\ - bcdD_{(\alpha)} - bR_{(\alpha 44\beta)}B^{(\beta)}\} + O_5. \end{aligned} \quad (125)$$

To avoid confusion with the formulae which follow, we note that, although the principal normals have been introduced, the coordinates $X_{(\alpha)}$ are tied to a Fermi triad.

We shall now use the principal vectors of C as a frame of reference,

taking in turn

$$\lambda^i = B^i, \quad \lambda^i = C^i, \quad \lambda^i = D^i. \quad (126)$$

In this frame of reference, the coordinates of P' are

$$Y_{(1)} = -\Omega_i B^i, \quad Y_{(2)} = -\Omega_i C^i, \quad Y_{(3)} = -\Omega_i D^i. \quad (127)$$

To calculate these coordinates for the falling apple, we have to go back to the equations (115)–(118) and substitute for λ^i from (126). Thus, using labels as in (127), we get

$$\begin{aligned} [D^2\phi]_{(1)} &= B^i D A_i = b, \\ [D^3\phi]_{(1)} &= 3DB^i D A_i + B^i D^2 A_i = Db, \\ [D^4\phi]_{(1)} &= D^2b + 9b^3 - 3bc^2 + bR_{(1441)}; \end{aligned} \quad (128)$$

$$\begin{aligned} [D^2\phi]_{(2)} &= 0, \\ [D^3\phi]_{(2)} &= -2bc, \\ [D^4\phi]_{(2)} &= -5bDc - 2cDb + bR_{(2441)}; \end{aligned} \quad (129)$$

$$\begin{aligned} [D^2\phi]_{(3)} &= 0, \\ [D^3\phi]_{(3)} &= 0, \\ [D^4\phi]_{(3)} &= 3bcd + bR_{(3441)}. \end{aligned} \quad (130)$$

In the R -terms, the components of the Riemann tensor are taken on the tetrad (B^i, C^i, D^i, A^i) in order. We have then, remembering the minus signs in (127),

$$\begin{aligned} Y_{(1)} &= -\frac{1}{2}bs^2 - \frac{1}{6}s^3Db + \frac{1}{24}s^4(3bc^2 - 9b^3 - D^2b - bR_{(1441)}) + O_5, \\ Y_{(2)} &= \frac{1}{3}s^3bc + \frac{1}{24}s^4(5bDc + 2cDb - bR_{(2441)}) + O_5, \\ Y_{(3)} &= -\frac{1}{24}s^4(3bcd + bR_{(3441)}) + O_5. \end{aligned} \quad (131)$$

These formulae express the fall of the apple in terms of coordinates based on the principal normals of the world-line of the branch of the tree.

The interpretation of (131) is simpler than that of (123) or (125) because the three Y 's are of different orders of magnitude. If we neglect O_4 , we see a motion primarily in the sense of the vector $-B^i$ with a small deviation (O_3) in the direction of C^i . In the representative 3-

space in which B^i , C^i , D^i are taken as axes of coordinates the trajectory is as illustrated in Fig. 14; its equation is

$$Y_{(2)}^2 = -\frac{8}{9} \frac{c^2}{b} Y_{(1)}^3. \quad (132)$$

We note that the initial acceleration is b in the direction $-B^i$.

These results throw light on the world-line of a terrestrial observer. If he drops an apple, its initial acceleration is vertically down (the direction of a plumb line) and its magnitude is g . Comparing experiment with relativistic theory, we see that *the first normal to the world-line of a terrestrial observer points vertically up (against the plumb line) and the first curvature of his world-line is g ; on the equator*

$$b = g = 978.05 \text{ cm sec}^{-2} = 3.263 \times 10^{-8} \text{ sec}^{-1}. \quad (133)$$

In writing down this last numerical value, we are to remember that throughout this book the fundamental measurement is the measurement of time, and for that we conveniently use the *second*, which we take to be a conventional multiple of the period of (say) the cadmium red line, the multiple being that which agrees with the best experimental value available. From the experimental value of the speed of light we can then assign a value to the *centimetre*: we have

$$\begin{aligned} 1 \text{ sec} &= 2.998 \times 10^{10} \text{ cm}, \\ 1 \text{ cm} &= 3.336 \times 10^{-11} \text{ sec}. \end{aligned} \quad (134)$$

Using this last value, we obtain the final number in (133). For the radius of curvature of the terrestrial observer's world-line we have

$$b^{-1} = 3.065 \times 10^7 \text{ sec}, \quad (135)$$

which is roughly 1 year¹.

When the rotation of the earth is taken into account in Newtonian mechanics, we find that the trajectory of a falling body deviates towards the east, and if we depict that trajectory in Fig. 14, with B^i pointing

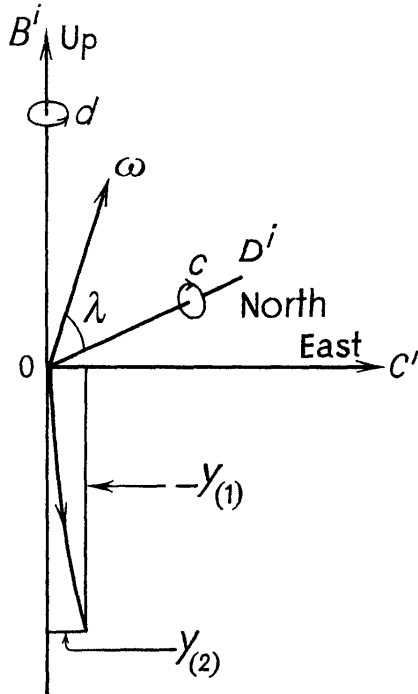


Fig. 14 – Free fall viewed relative to the triad of principal normals of the world-line of the support

¹ A list of numerical values will be found in Appendix B.

vertically upward and C^i pointing east, then its equation agrees with (132) if we write

$$\frac{c^2}{b} = \frac{\omega^2 \cos^2 \lambda}{g}, \quad (136)$$

where ω is the angular velocity of the earth and λ the latitude of the place¹. Thus we see that *the second normal to the world-line of a terrestrial observer points to the east and the second curvature of his world-line is*

$$c = \omega \cos \lambda. \quad (137)$$

For an observer on the equator, the numerical values of this second curvature and the corresponding radius of curvature are

$$c = 7.292 \times 10^{-5} \text{ sec}^{-1}, \quad c^{-1} = 1.371 \times 10^4 \text{ sec.} \quad (138)$$

This radius is about 4 hours.

It is interesting that, although the acceleration due to gravity is much more important in ordinary mechanics than the effects of the earth's rotation, the first curvature b is actually much smaller than the second curvature c ; in fact, we have

$$b/c = 4.475 \times 10^{-4}. \quad (139)$$

We have seen that, by observation of the acceleration and deflection of a falling body, it is possible to identify physically the normals B^i , C^i (and hence, by orthogonality, D^i) of the observer's world-line, and also the first two curvatures b , c . It does not appear from (131) that the third curvature d is readily accessible to dynamical observation. However, there is another approach. In § 8 we were able to identify Fermi transport physically by means of the bouncing photon, and we shall now see that the value of d follows from the connection between the triad (B^i, C^i, D^i) and the Fermi triad $\lambda_{(\alpha)}^i$. The following work down to (145) inclusive is pure geometry in the manner of I-§§ 3,4.

Let us resolve the unit normals on the Fermi triad:

$$B^i = B_{(\alpha)} \lambda_{(\alpha)}^i, \quad C^i = C_{(\alpha)} \lambda_{(\alpha)}^i, \quad D^i = D_{(\alpha)} \lambda_{(\alpha)}^i. \quad (140)$$

To guide our thoughts, we think of $\lambda_{(\alpha)}^i$ as rectangular Cartesian axes;

¹ Cf. J. L. SYNGE and B. A. GRIFFITH, *Principles of Mechanics* (3rd Edn., McGraw-Hill, New York, 1959), p. 364. We here assume that the actual deviation agrees with the Newtonian formula. N. A. Kozyrev has recently claimed that it does not agree.

then we see three mutually orthogonal unit Cartesian vectors $B_{(\alpha)}$, $C_{(\alpha)}$, $D_{(\alpha)}$ — that is how the normals appear to a Fermi observer. We substitute (140) into the Frenet-Serret formulae 1-(55), and use the equation of Fermi transport,

$$\frac{\delta \lambda_{(\alpha)}^i}{\delta s} = b A^i \lambda_{(\alpha)}^j B_j = b A^i B_{(\alpha)}. \quad (141)$$

We get, with a prime for d/ds ,

$$B'_{(\alpha)} = c C_{(\alpha)}, \quad C'_{(\alpha)} = d D_{(\alpha)} - c B_{(\alpha)}, \quad D'_{(\alpha)} = -d C_{(\alpha)}. \quad (142)$$

Now if, in Euclidean kinematics, an orthonormal triad ($\mathbf{i}, \mathbf{j}, \mathbf{k}$) rotates with angular velocity $\boldsymbol{\omega}$, we have

$$\mathbf{i}' = \boldsymbol{\omega} \times \mathbf{i} = \boldsymbol{\omega} \times (\mathbf{j} \times \mathbf{k}) = \omega_3 \mathbf{j} - \omega_2 \mathbf{k}, \quad (143)$$

and two similar equations, where $\omega_1, \omega_2, \omega_3$ are the components of $\boldsymbol{\omega}$ on the triad. Hence, if $\omega_1, \omega_2, \omega_3$ are the components of angular velocity of the triad of normals relative to the Fermi triad, with these components taken along the normals, we have

$$\begin{aligned} B'_{(\alpha)} &= \omega_3 C_{(\alpha)} - \omega_2 D_{(\alpha)}, \\ C'_{(\alpha)} &= \omega_1 D_{(\alpha)} - \omega_3 B_{(\alpha)}, \\ D'_{(\alpha)} &= \omega_2 B_{(\alpha)} - \omega_1 C_{(\alpha)}. \end{aligned} \quad (144)$$

On comparing these equations with (142), we obtain the following simple expressions in terms of the second and third curvatures of the timelike world-line on which the triads are given:

$$\omega_1 = d, \quad \omega_2 = 0, \quad \omega_3 = c. \quad (145)$$

These components of angular velocity are indicated in Fig. 14.

To interpret this angular velocity in the case of a terrestrial observer, we have reason to regard a Fermi triad as a triad without rotation in the ordinary sense. Then the angular velocity of (145) is to be identified with the angular velocity of the earth (marked ω in Fig. 14). Then, λ being the latitude of the observer, we have

$$c = \omega \cos \lambda, \quad d = \omega \sin \lambda. \quad (146)$$

The first of these is the same as (137), obtained dynamically. The second is new: it gives us the third curvature of the world-line of an observer on the earth. It is interesting that c and d are in general of

the same order of magnitude; however, c dominates at the equator and d at the poles.

This physical identification of the curvatures of the world-line of a terrestrial observer is, of course, somewhat crude, because the earth's orbital motion has not been considered. Anyone who wishes to examine the question more carefully should remember that (145) is an exact mathematical formula, independent of any physical interpretation.

§ 10. THE BALLISTIC SUICIDE PROBLEM

The usual problem of ballistics is to aim a projectile so that it hits someone else. Here we consider the ballistic suicide problem: the projectile is to hit the projector himself!

However perverse such a problem may be sociologically, it is a neat problem in relativity, because there are only two observations and both are made by the same observer. Moreover it forces us to realize that although the trajectory of a projectile fired straight upwards seems sharply curved at the top, from a space-time standpoint it is as straight as possible (geodesic).

Fig. 15 shows the world line C of the suicidal observer: Q_1 is the event of the projectile's departure, Q_2 the event of its return, and the geodesic Γ joining Q_1 and Q_2 is the history of the projectile. We apply the argument of II-§ 13, taking the point Q_0 of Fig. II-15 halfway between Q_1 and Q_2 , so that we may write

$$s_1 = -\frac{1}{2}\sigma, \quad s_2 = \frac{1}{2}\sigma, \quad s_2 - s_1 = \sigma. \quad (147)$$

Then σ is the time-interval between Q_1 and Q_2 (the projectile's time of flight) as recorded by a clock carried by the observer. In accordance with II-(258), we denote by τ the other measure of the time of flight, viz. that recorded by a clock carried on the projectile. As in II-(259), we have then

$$\tau = \sigma + \frac{1}{24}b^2\sigma^3 + O_4, \quad (148)$$

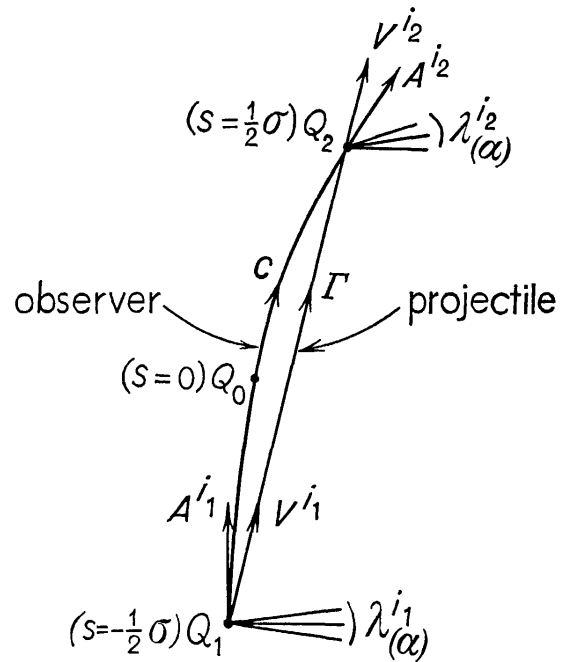


Fig. 15 – Ballistic suicide

where b is the first curvature of C at Q_0 and σ is small (O_1).

We note that $\tau > \sigma$, i.e. the projectile's clock runs fast when compared with the observer's¹. It is interesting to see how high a projectile must go above the earth's surface to give a time-difference within the limits of modern chronometry². To make

$$\tau - \sigma \geq 10^{-10} \text{ sec}, \quad (149)$$

we need to take σ at least as great as the value given by

$$b^2 \sigma^3 = 24 \times 10^{-10} \text{ sec}. \quad (150)$$

For a terrestrial observer, we take b as in (133); then we get

$$\sigma^3 = 2.254 \times 10^6 \text{ sec}^3, \quad \sigma = 131 \text{ sec}, \quad (151)$$

which is a time of flight of somewhat over 2 minutes. This corresponds to a trajectory rising to a height of $\frac{1}{8}g\sigma^2 = 21 \text{ km}$ roughly. It is hardly necessary to point out that in seeking conceptual simplicity we have wandered rather far from reality in eliminating the resistance of the air; its inclusion would, however, involve some rather ugly complications.

Let us now consider the projectile's initial and final velocities, measured on a Fermi triad $\lambda_{(\alpha)}^i$ carried on the observer's world-line C . If the projectile's 4-velocity is V^i and its components on the Fermi triad are $V_{(\alpha)}$, then this 4-velocity is connected accurately with the first derivatives of the world-function $\Omega(Q_1Q_2)$ by [cf. II-(17)]

$$\begin{aligned} \Omega_{i_1} &= -\tau V_{i_1}, & \Omega_{i_2} &= \tau V_{i_2}, \\ \Omega_{(\alpha_1)} &= -\tau V_{(\alpha_1)}, & \Omega_{(\alpha_2)} &= \tau V_{(\alpha_2)}. \end{aligned} \quad (152)$$

But by II-(269) we have

$$\begin{aligned} \Omega_{(\alpha_1)} &= -\frac{1}{2}\sigma^2\{(DA)_{(\alpha)} - \frac{1}{6}\sigma(D^2A)_{(\alpha)}\} + O_4, \\ \Omega_{(\alpha_2)} &= -\frac{1}{2}\sigma^2\{(DA)_{(\alpha)} + \frac{1}{6}\sigma(D^2A)_{(\alpha)}\} + O_4, \end{aligned} \quad (153)$$

¹ This is an example of that basic fact of relativity which is often absurdly called the *clock paradox*.

² For the somewhat similar problem of red-shift for an artificial satellite, see S. F. SINGER [1956], B. HOFFMANN [1957], A. DAS [1957b].

where the A -terms are evaluated at Q_0 . Hence, by (148) and (152),

$$\begin{aligned} V_{(\alpha_1)} &= \frac{1}{2}\sigma(DA)_{(\alpha)} - \frac{1}{12}\sigma^2(D^2A)_{(\alpha)} + O_3, \\ V_{(\alpha_2)} &= -\frac{1}{2}\sigma(DA)_{(\alpha)} - \frac{1}{12}\sigma^2(D^2A)_{(\alpha)} + O_3, \end{aligned} \quad (154)$$

these being respectively the initial and final velocities.

In first approximation, we have

$$V_{(\alpha_1)} = \frac{1}{2}\sigma b B_{(\alpha)} + O_2, \quad V_{(\alpha_2)} = -\frac{1}{2}\sigma b B_{(\alpha)} + O_2, \quad (155)$$

so that the projectile is thrown along the first normal to C , and returns along the first normal. Now for a quick terrestrial suicide, we know that the projectile should be thrown straight upwards with a speed $\frac{1}{2}g\sigma$, where σ is the time of flight, and so we verify what we found in § 9 — the first normal to the terrestrial observer's world-line points vertically up and its first curvature is $b = g$, as in (133).

But the final velocity is not precisely the reverse of the initial velocity, for by (154) and (124) we have

$$\begin{aligned} V_{(\alpha_1)} + V_{(\alpha_2)} &= -\frac{1}{6}\sigma^2(D^2A)_{(\alpha)} + O_3 \\ &= -\frac{1}{6}\sigma^2(DbB_{(\alpha)} + bcC_{(\alpha)}) + O_3, \end{aligned} \quad (156)$$

where $Db = db/ds$ at Q_0 . Referring to Fig. 14, we see that in the terrestrial case this 3-vector has a component $\frac{1}{6}\sigma^2 Db$ down (direction of $-B^i$) and a component $\frac{1}{6}\sigma^2 bc$ westward (direction of $-C^i$).

Thus we see that ballistic suicide observations provide in principle the first and second normals (B^i , C^i) of the observer's world-line, and also b , c , db/ds , the observed quantities being the flight-time σ and the 3-vectors in (155) and (156). As regards numerical values in the terrestrial case, we have b and c (equatorial) as in (133) and (138), and if we take σ as in (151), we find

$$\frac{1}{2}\sigma b = 2.137 \times 10^{-6}, \quad \frac{1}{6}\sigma^2 bc = 6.805 \times 10^{-9}. \quad (157)$$

These terms in (155), (156) are dimensionless, and are in fact fractions of the speed of light. In more usual units, we have

$$\frac{1}{2}\sigma b = 6.41 \times 10^4 \text{ cm sec}^{-1}, \quad \frac{1}{6}\sigma^2 bc = 204 \text{ cm sec}^{-1}. \quad (158)$$

These figures give an idea of the magnitudes involved. We must of course remember that we are here committed to the method of II-§ 13 which uses power series, and the method is applicable only to a fairly restricted domain of space-time. Suicide by means of a projectile which travelled to the vicinity of the moon, for example, and then returned to the earth would be quite beyond the scope of this method.

§ 11. STATICAL MEASUREMENT OF GRAVITATIONAL FIELDS

As was shown in § 9, the observation of a falling body gives information about the world-line of the observer, but it seems to throw little light on the strength of the gravitational field, i.e. the components of the Riemann tensor R_{ijkl} . True, these components occur in (131), but they are masked by other terms.

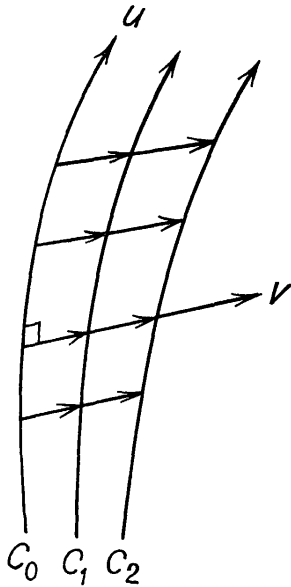


Fig. 16 – Statical measurement of gravitational field

We now consider the possibility of measuring a gravitational field by experiments which we may call *statical* by analogy with the familiar statical experiment in which two bodies each of mass m are weighed against one another, with one body in the attic and the other in the basement. In Newtonian physics, the difference between the two weights, divided by the difference in height, and by m , gives the rate of change of the intensity of the gravitational field. We shall investigate what such a quotient represents in relativity.

In Fig. 16, C_0 is the world-line of an observer. If he drops an object, its geodesic world-line departs from C_0 , but as long as he holds the object, its world-line coincides with C_0 . If A^i ($= dx^i/ds$) is the 4-velocity of a particle of mass m carried on C_0 , the 4-force acting on it is

$$F^i = m \frac{\delta A^i}{\delta s} = mbB^i, \quad (159)$$

where b is the first curvature of C_0 and B^i its first unit normal. In view of the considerable differences of opinion among Newtonian physicists as to the logical status of the concept of force, it would be foolish to insist pedantically here either (i) that (159) is the *definition* of force, or (ii) that (159) is a *law of motion* involving an already accepted concept of 4-force.

However we look at it, we may regard F^i as statically measurable (with a spring-balance for example), and so the observer can obtain statically the vector B^i and the curvature b . Alternatively, the observer may rely on dynamical experiments, observing a falling body, and indeed this method is more powerful, for (as seen in § 9) it gives not only B^i and b but also the other normals C^i , D^i and the other

curvatures c, d . We shall proceed on this dynamical basis because it is richer in results, but we can always go back to the more restricted statical results by putting $\gamma = \delta = 0$ in the formulae which follow.

At each event on C_0 we have the orthogonal triad of normals (B^i, C^i, D^i) . Let β, γ, δ be any constants satisfying

$$\beta^2 + \gamma^2 + \delta^2 = 1, \quad (160)$$

and let dv be an infinitesimal constant. Then the vector

$$(\beta B^i + \gamma C^i + \delta D^i)dv \quad (161)$$

can be constructed at each event on C_0 , and the extremities of these infinitesimal vectors give an adjacent world-line C_1 , which may be used for observations by a second observer. Stepping off from C_1 along the vector (161), with $\beta, \gamma, \delta, dv$ the same constants as before, we get a third adjacent curve C_2 . Repeating indefinitely, we obtain a single infinity of world-lines, which form a 2-space, and on this 2-space we take parameters (u, v) as follows. Let $u = s$ on C_0 and $u = \text{const.}$ on each of the curves tangent to the vectors (161) (Fig. 16). Let $v = \text{const.}$ on each of the world-lines C_0, C_1, C_2, \dots with $v = 0$ on C_0 . Writing

$$U^i = \frac{\partial x^i}{\partial u}, \quad V^i = \frac{\partial x^i}{\partial v}, \quad (162)$$

we have then the following formulae:

$$\begin{aligned} \frac{\delta U^i}{\delta v} &= \frac{\delta V^i}{\delta u}, \\ V^i &= \beta B^i + \gamma C^i + \delta D^i, \quad V_i V^i = 1, \\ U_i V^i &= 0, \quad A^i = U^i/U, \quad U^2 = -U_i U^i, \quad U > 0, \\ (U)_{v=0} &= 1. \end{aligned} \quad (163)$$

In these formulae A^i, B^i, C^i, D^i are the unit tangent and unit normals to any one of the curves $v = \text{const.}$

Each of the world-lines $v = \text{const.}$ has a first curvature $b(u, v)$. Our object is to calculate $(\partial b / \partial v)_{v=0}$, an observable relativistic invariant which corresponds to the rate of change with height of the Newtonian gravitational field, as described above.

At any point of the 2-space of world-lines, we have

$$bB^i = \frac{\delta A^i}{\delta s} = \frac{1}{U} \frac{\delta}{\delta u} \left(\frac{U^i}{U} \right), \quad (164)$$

since $ds/du = U$. Hence

$$\begin{aligned} bB^i &= \frac{1}{U^2} \frac{\delta U^i}{\delta u} - \frac{U^i}{U^3} \frac{\partial U}{\partial u}, \\ b^2 U^4 &= \frac{\delta U^i}{\delta u} \frac{\delta U_i}{\delta u} + \left(\frac{\partial U}{\partial u} \right)^2, \end{aligned} \quad (165)$$

since $U \partial U / \partial u = -U_i \delta U^i / \delta u$. Differentiating with respect to v , we get

$$U^4 b \frac{\partial b}{\partial v} + 2b^2 U^3 \frac{\partial U}{\partial v} = \frac{\delta U^i}{\delta u} \frac{\delta^2 U_i}{\delta v \delta u} + \frac{\partial U}{\partial u} \frac{\partial^2 U}{\partial v \partial u}. \quad (166)$$

Now, by (163) and I-(95),

$$\begin{aligned} U \frac{\partial U}{\partial v} &= -U_i \frac{\delta U^i}{\delta v} = -U_i \frac{\delta V^i}{\delta u}, \\ \frac{\delta^2 U_i}{\delta v \delta u} &= \frac{\delta^2 V_i}{\delta u^2} + R_{ijkm} U^j V^k U^m, \end{aligned} \quad (167)$$

and so, putting $v = 0$ in (166) and noting that then $u = s$, $U = 1$, $\partial U / \partial u = 0$, we obtain

$$b \frac{\partial b}{\partial v} - 2b^2 A^i \frac{\delta V_i}{\delta s} = \frac{\delta A^i}{\delta s} \left(\frac{\delta^2 V_i}{\delta s^2} + R_{ijkm} A^j V^k A^m \right) \quad (168)$$

or

$$\frac{\partial b}{\partial v} = 2b A^i \frac{\delta V_i}{\delta s} + B^i \left(\frac{\delta^2 V_i}{\delta s^2} + R_{ijkm} A^j V^k A^m \right). \quad (169)$$

By (163) and I-(55) we have

$$\begin{aligned} \frac{\delta V_i}{\delta s} &= \beta(cC_i + bA_i) + \gamma(dD_i - cB_i) - \delta dC_i, \\ \frac{\delta^2 V_i}{\delta s^2} &= \beta\{c'C_i + c(dD_i - cB_i) + b'A_i + b^2 B_i\} \\ &\quad + \gamma\{d'D_i - d^2 C_i - c'B_i - c(cC_i + bA_i)\} \\ &\quad - \delta\{d'C_i + d(dD_i - cB_i)\}, \end{aligned} \quad (170)$$

where the prime means d/ds ; hence

$$\begin{aligned} A^i \frac{\delta V_i}{\delta s} &= -\beta b, \\ B^i \frac{\delta^2 V_i}{\delta s^2} &= \beta(b^2 - c^2) - \gamma c' + \delta cd. \end{aligned} \quad (171)$$

We substitute these values in (169), and use (1, 2, 3, 4) as labels for components on the orthonormal tetrad (B^i, C^i, D^i, A^i) , in this order. Thus we obtain (for $v = 0$, i.e. on C_0)

$$\frac{\partial b}{\partial v} = \beta(R_{(1414)} - b^2 - c^2) + \gamma(R_{(1424)} - c') + \delta(R_{(1434)} + cd). \quad (172)$$

If we now put in turn each of constants β, γ, δ equal to unity and the other two zero, we get the following formulae for the rates of change of b in the directions of the normals B^i, C^i, D^i in order:

$$\begin{aligned} \left(\frac{\partial b}{\partial v} \right)_{(1)} &= R_{(1414)} - b^2 - c^2, \\ \left(\frac{\partial b}{\partial v} \right)_{(2)} &= R_{(1424)} - c', \\ \left(\frac{\partial b}{\partial v} \right)_{(3)} &= R_{(1434)} + cd. \end{aligned} \quad (173)$$

Since dv is in fact an element of distance, we see that *these formulae give the rates of change with distance in the tension in a plumb line of unit mass for displacements in the directions of the three normals of the observer's world-line.*

In the terrestrial model (Fig. 14) these three directions are respectively (i) vertically up (as indicated by a plumb line), (ii) to the east, and (iii) to the north. By comparison with a Newtonian model, we shall evaluate, at any point on the surface of a rotating body, the components of the Riemann tensor which occur in (173).

Fig. 17 shows a quadrant of the axial section of a body rotating with angular velocity ω . The observer is at O , at latitude λ . The unit vectors, which we now indicate by **B**, **C**, **D**, point as shown, and we take them as axes $Oxyz$. ON is dropped perpendicularly on the axis of

rotation, the direction of which is indicated by the unit vector \mathbf{K} ; let $\overrightarrow{NO} = \rho_0$.

Let \mathbf{g} be the vector representing the tension (from support to bob) in a short plumb line of unit mass attached to a point P with coordinates (x, y, z) . Let $V(x, y, z)$ be the Newtonian gravitational

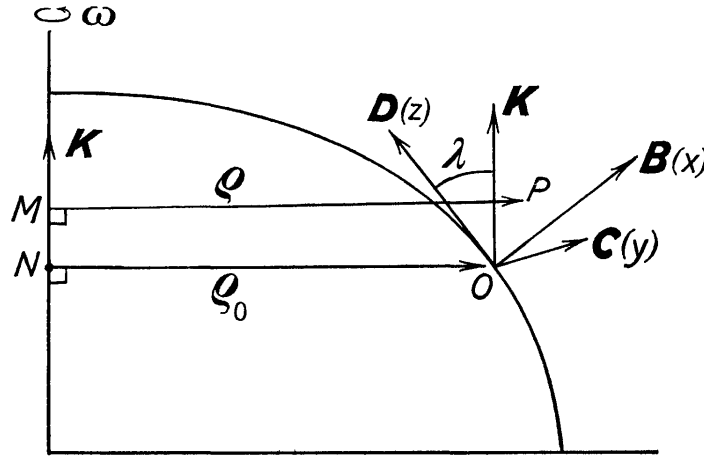


Fig. 17 – Rotating body

potential, chosen with that sign which makes gravitational intensity equal to ∇V . Then

$$\mathbf{g} = \nabla V + \rho \omega^2, \quad (174)$$

where $\rho = \overrightarrow{MP}$, perpendicular to \mathbf{K} . Now

$$\rho = \rho_0 + \mathbf{r} - \mathbf{K}(\mathbf{r} \cdot \mathbf{K}), \quad (175)$$

where

$$\mathbf{r} = \overrightarrow{OP} = x\mathbf{B} + y\mathbf{C} + z\mathbf{D}. \quad (176)$$

Therefore

$$\frac{\partial \rho}{\partial x} = \mathbf{B} - \mathbf{K}(\mathbf{B} \cdot \mathbf{K}) = \mathbf{B} - \mathbf{K} \sin \lambda = \mathbf{B} \cos^2 \lambda - \mathbf{D} \sin \lambda \cos \lambda,$$

$$\frac{\partial \rho}{\partial y} = \mathbf{C}, \quad (177)$$

$$\frac{\partial \rho}{\partial z} = \mathbf{D} - \mathbf{K}(\mathbf{D} \cdot \mathbf{K}) = \mathbf{D} - \mathbf{K} \cos \lambda = -\mathbf{B} \sin \lambda \cos \lambda + \mathbf{D} \sin^2 \lambda.$$

The magnitude g of the vector \mathbf{g} is the quantity b which occurs in

(173), and we seek to evaluate $\partial g/\partial x$, $\partial g/\partial y$, $\partial g/\partial z$ at O . Now

$$g \frac{\partial g}{\partial x} = \mathbf{g} \cdot \frac{\partial \mathbf{g}}{\partial x}, \text{ etc.}, \quad (178)$$

and at O we have

$$(\mathbf{g})_0 = -g_0 \mathbf{B}, \quad (179)$$

since \mathbf{B} is vertical by plumb line; here g_0 is the tension in a plumb line at O . Hence, by (174) and (178),

$$\begin{aligned} \left(\frac{\partial g}{\partial x} \right)_0 &= -\mathbf{B} \cdot \left(\frac{\partial \mathbf{g}}{\partial x} \right)_0 = -V_{xx} - \omega^2 \cos^2 \lambda, \\ \left(\frac{\partial g}{\partial y} \right)_0 &= -\mathbf{B} \cdot \left(\frac{\partial \mathbf{g}}{\partial y} \right)_0 = -V_{xy} = 0, \\ \left(\frac{\partial g}{\partial z} \right)_0 &= -\mathbf{B} \cdot \left(\frac{\partial \mathbf{g}}{\partial z} \right)_0 = -V_{xz} + \omega^2 \sin \lambda \cos \lambda, \end{aligned} \quad (180)$$

the partial derivatives on the right being evaluated at O . The zero value in the second equation arises from the fact that $V_y = 0$ all over the plane $y = 0$. Comparing (173) and (180), we get

$$\begin{aligned} R_{(1414)} - b^2 - c^2 &= -V_{xx} - \omega^2 \cos^2 \lambda, \\ R_{(1424)} - c' &= 0, \\ R_{(1434)} + cd &= -V_{xz} + \omega^2 \sin \lambda \cos \lambda. \end{aligned} \quad (181)$$

Recalling the values found for c and d in (146), certain terms cancel, and (since $c' = 0$) we get

$$R_{(1414)} = -V_{xx} + g_0^2, \quad R_{(1424)} = 0, \quad R_{(1434)} = -V_{xz}. \quad (182)$$

Thus we have evaluated certain components of the Riemann tensor in terms of second derivatives of the Newtonian potential, on the assumption (a pretty sound one) that Newtonian mechanics gives a very good approximation to physical reality; or, more cautiously, we might say that the evaluation (182) is valid if the body and its rotation are such that Newtonian mechanics is valid. Note that $R_{(1414)}$ is the negative of the Riemannian curvature for the 2-element defined by the observer's world-line and its first normal.

To estimate the components (182) at a point on the surface of the earth, let us take the earth to be a homogeneous sphere of radius a ; then, by Newtonian theory,

$$V_{xx} = 2\bar{g}/a, \quad V_{xz} = 0, \quad (183)$$

where \bar{g} is the acceleration due to gravity alone. The difference between g_0 and \bar{g} is due to the earth's rotation, but it is small and we may take

$$\begin{aligned} \bar{g} &= g_0 = 980 \text{ cm sec}^{-2} = 3.27 \times 10^{-8} \text{ sec}^{-1}, \\ a &= 6.37 \times 10^8 \text{ cm} = 2.12 \times 10^{-2} \text{ sec}. \end{aligned} \quad (184)$$

Since

$$g_0^2/V_{xx} = \frac{1}{2}\bar{g}a = 3.47 \times 10^{-10}, \quad (185)$$

the term g_0^2 in (182) may be neglected, and, of the components listed there, the only survivor is

$$R_{(1414)} = -2\bar{g}/a = -3.08 \times 10^{-6} \text{ sec}^{-2}. \quad (186)$$

This corresponds to a 'radius of curvature' of 570 sec, which is of the same order as the radius of the earth's orbit (see Appendix B).

§ 12. FERMI-WALKER TRANSPORT ALONG A SPACELIKE CURVE AND ITS PHYSICAL MEANING

Fermi-Walker (F-W) transport along a timelike curve in space-time was defined in I-(72), and Fermi transport in I-(84). A physical interpretation of this type of transport was given in § 8 in terms of the bouncing photon: the criterion for the Fermi transport of a frame of reference is that, relative to the frame, the directions of emission and return of the bouncing photon are the same to the first order in the distance of the object from which the photon bounces.

A spacelike curve in space-time is physically much less familiar than a timelike curve; the latter is the history of a moving particle, but the former can only be described negatively as a set of events between which causal relationships are impossible, or as a set of events which cannot represent the history of a photon or a material particle. However, as far as the mathematics of F-W and Fermi transport is concerned, there is not much difference. We simply change the sign in the formulae of I-§ 4 and adopt the following definitions for a spacelike curve C :

$$\text{Fermi-Walker transport: } DF^i = -F_j(A^i DA^j - A^j DA^i), \quad (187)$$

$$\text{Fermi transport: } DF^i = -A^i F_j DA^j, \quad (188)$$

where $D = \delta/\delta s$ (absolute differentiation with respect to the spatial measure of C) and $A^i = dx^i/ds$, the unit tangent to C , satisfying

$$A_i A^i = 1. \quad (189)$$

It is easy to verify that A^i itself satisfies (187) and that scalar products are conserved.

To discuss the physical meaning of F-W transport along a space-like curve, we may use the same type of argument as that of § 8 (bouncing photon), making the necessary changes in sign. However, it would be tedious to go through the details, and it will suffice to explain the essential changes in the physical situation and mention the most important results.

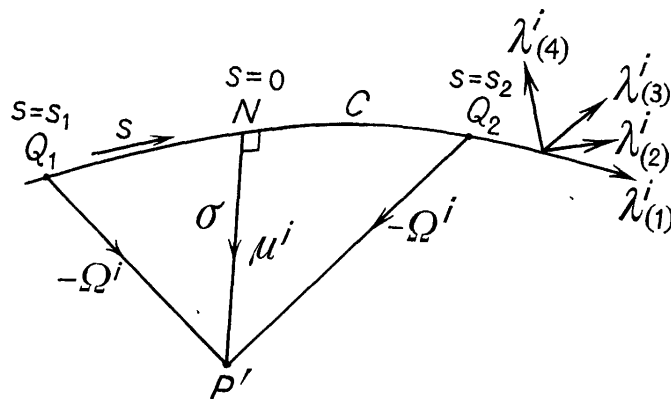


Fig. 18 – Physical meaning of F-W transport along a spacelike curve

Fig. 18 is Fig. 12 turned on its side¹. C is a spacelike curve and on it s is spatial measure as given by the Riemannian metric (ds has a chronometric meaning, cf. § 4). On C we take an orthonormal tetrad $\lambda^i_{(a)}$, with $\lambda^i_{(1)}$ tangent to C and $\lambda^i_{(4)}$ timelike. Let Q_1 and Q_2 be two adjacent events on C . We seek a physical test to determine whether the tetrads $\lambda^i_{(a)}$ at Q_1 and Q_2 are in fact consistent with the condition of F-W transport.

To make such a test, we think of some event P' which is such that, if an explosion occurs at P' , a photon from the explosion passes through Q_1 and another through Q_2 . In fact, two observers whose world lines (not shown) pass respectively through Q_1 and Q_2 see the explosion at those events. The null geodesic $P'Q_1$ defines a direction at Q_1 relative to the triad $\lambda^i_{(\alpha)}$ there, with direction cosines $\theta_{(\omega)}(Q_1)$, say, and likewise $P'Q_2$ has direction cosines $\theta_{(\omega)}(Q_2)$ at Q_2 .

¹ Throughout this book space-time diagrams show future-pointing timelike vectors pointing up the page, making with the vertical an angle less than 45° .

It is clear that, although there is now no bouncing photon, an investigation of these direction cosines will be very similar to the investigation of direction cosines in § 8. We draw $P'N$, a geodesic orthogonal to C , and put $\sigma = P'N$, $s = 0$ at N , $s = s_1$ at Q_1 , and $s = s_2$ at Q_2 , just as we did in § 8. Then, assuming σ small (O_1) we find, using the world-function Ω and pursuing the same course as in § 8,

$$s_2 - s_1 = 2\sigma + O_2, \quad (190)$$

and, *provided the reference tetrad undergoes F-W transport with* $\lambda_{(1)}^i = A^i$,

$$\begin{aligned} \theta_{(1)}(Q_1) + \theta_{(1)}(Q_2) &= O_2, \\ \theta_{(2)}(Q_1) - \theta_{(2)}(Q_2) &= O_2, \\ \theta_{(3)}(Q_1) - \theta_{(3)}(Q_2) &= O_2. \end{aligned} \quad (191)$$

(Note the $+$ sign in the first equation.) If the tetrad does not undergo F-W transport, then the right hand sides in (191) are O_1 . This gives us a physical test for F-W transport: the criterion is that, if σ is so small that O_2 is negligible, then *the direction cosine relative to the tangent is reversed in sign, and the other two are unchanged*.

Since F-W transport appears to be a most fundamental physical operation (it may be said to represent the absence of rotation), we would like to get some intuitive insight into the above test. This is not easy because Newtonian ideas are so deeply ingrained in us that even the idea of a spacelike curve in space-time does not come easily to us. When we seek to understand difficult ideas, fantastic illustrations are best, and the following may be of assistance.

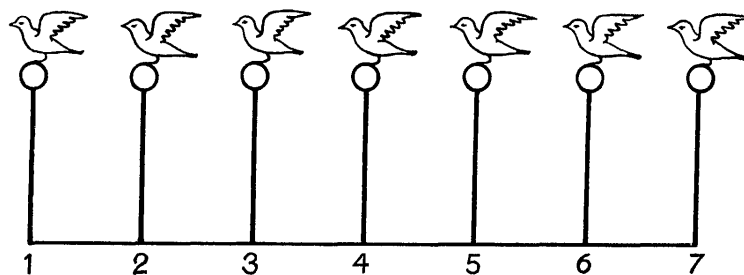


Fig. 19 – The birds on the lamp-posts

Imagine (Fig. 19) a row of equidistant lamp-posts, numbered 1, 2, 3, On each lamp-post there sits a bird. The lamps are out, but each bird is capable of lighting the lamp on which he sits.

Suppose first that Bird No. 1 lights his lamp, that Bird No. 2 lights his lamp when he *sees* the light from No. 1, that Bird No. 3 lights his

when he *sees* the light of No. 2, and so on. Then the lighting of the lamps forms a set of events which may be regarded as a curve C in space-time, and C is a *null curve*.

Now suppose that the birds follow the same plan, but that they are a little slow in their reactions, so that there is a delay between seeing and acting. Now C becomes timelike, and Bird No. 1 could take off and keep pace with the lightings, i.e. make C his world-line.

Lastly, suppose that the birds light their lamps without any causal chain connecting them. This might happen in various ways. They might light their lamps when they saw the sun set, for example, with the sun away behind the picture. Then, it might be impossible for any bird to leave his lamp immediately after lighting it and reach the next lamp before it was lit. In that case C would be spacelike, and that is the case we have to consider.

Having constructed thus a spacelike curve C , the next step is to fit it with an orthonormal tetrad $\lambda_{(a)}^i$ — we do not bother about F-W transport yet. The tangent vector $\lambda_{(1)}^i$ is easy, for it is defined by the lightings of two adjacent lamps. But what of the timelike vector $\lambda_{(4)}^i$ orthogonal to C ? A timelike vector represents the 4-velocity of a particle, and the best plan is to let each bird take off into the air at the instant when he lights his lamp. But in what direction and how fast? Consider, for example, Bird No. 4. As he flies off he is to see simultaneously the lightings of Lamps Nos. 3 and 5. (Since C is spacelike, he does not see these lightings until after he lights his own lamp and takes off.) This plan does not, of course, fix $\lambda_{(4)}^i$ — it only restricts it. Any choice subject to the restriction having been made, it is an easy matter for each bird to define two vectors, $\lambda_{(2)}^i$ and $\lambda_{(3)}^i$, in his instantaneous space and orthogonal to $\lambda_{(1)}^i$.

We have now an orthonormal tetrad defined along the spacelike C , with $\lambda_{(1)}^i$ tangent to C and $\lambda_{(4)}^i$ timelike and orthogonal to C . It remains to apply the test for F-W transport. Let us concentrate on Birds Nos. 3 and 4. We must arrange an explosion somewhere at some time such that the light-flash from it reaches these two birds at the instants when they light their lamps and take off. Each bird is then required to report the direction (relative to his triad $\lambda_{(\alpha)}^i$) in which he sees the flash of the explosion. The criterion for F-W transport is that the direction cosines relative to the tangent $\lambda_{(1)}^i$ should be equal in magnitude but opposite in sign, whereas the other direction cosines should be equal.

The reader is strongly advised to construct for himself other (perhaps simpler) illustrations. But he must avoid two traps. First, he must not allow the two Newtonian fallacies — absolute simultaneity and the rigid body. Secondly, since we are concerned with the general theory of relativity, the idea of a straight world-line with an infinite 3-flat orthogonal to it is not permitted; however, since we are working in the small and the Riemann tensor does not appear in our crude approximation, this is a more venial error.

In working with spacelike curves one is in danger at first of making colossal and lamentable errors such as identifying the spacelike curve with a stretched string. This is nonsense, because the curve is a 1-space and the history of the string is a 2-space. It is true that this 2-space is split up naturally into timelike curves (the histories of the particles which form the string), but it is not split up into spacelike curves; if we like, we can draw in the 2-space the orthogonal trajectories of the world-lines of the particles, and so obtain a set of spacelike curves each of which might be called a form of the string, but that is a somewhat arbitrary procedure. Steeped as we are in Newtonian ideas, it is necessary to emphasize, even *ad nauseam*, that space-time cannot, in general, be split into space and time in any invariant way. In a statical universe (see Chaps. VII and VIII) such a splitting does in fact occur, but that is a very special case. To understand space-time, it is best to leave statical universes out of account for the present.

In the argument which follows we shall obtain a physical interpretation of spacelike geodesics, but we shall do more than this: we shall attach physical meanings to the first normal B^i and the first curvature b of any spacelike curve.

In Fig. 20, C is a spacelike curve and U^i a timelike unit vector, orthogonal to C and carried along C by Fermi transport. In the directions U^i we draw timelike geodesics, thus forming a 2-space on which we put parameters (u, v) where u is geodesic distance from C and v is constant on each geodesic with $v = s$ on C , ds being the spatial element on C . Then, in the notation used frequently before,

$$\begin{aligned} U^i &= \frac{\partial x^i}{\partial u}, & V^i &= \frac{\partial x^i}{\partial v}, & \frac{\delta U^i}{\delta v} &= \frac{\delta V^i}{\delta u}, \\ U_i V^i &= 0, & \frac{\delta U^i}{\delta u} &= 0, & U_i U^i &= -1, \end{aligned} \tag{192}$$

and, since U^i undergoes Fermi transport on C , we have by (188)

$$\frac{\delta U^i}{\delta v} = -V^i U_j \frac{\delta V^j}{\delta v} \text{ for } u = 0, \quad (193)$$

for on C we have $V^i = A^i = dx^i/ds$, the unit tangent to C .

The general idea now is this: the more C is curved, the more will the geodesics $v = \text{const.}$ tend to close in on the convex side of C , and

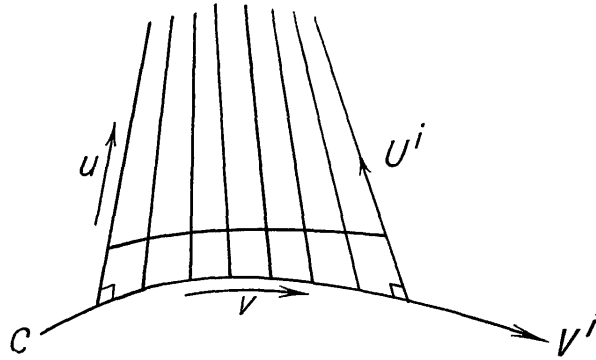


Fig. 20 – Physical meaning of first normal and curvature of spacelike curve C . (The diagram is not carelessly drawn; for C convex on top, the geodesics normal to it have a tendency to *converge*)

the rate of decrease of the separation between neighbouring geodesics gives a measure of the curvature of C . We have

$$\frac{1}{2} \frac{\partial}{\partial u} (V_i V^i) = V_i \frac{\delta V^i}{\delta u} = V_i \frac{\delta U^i}{\delta v}, \quad (194)$$

and, by (193), this gives, for $u = 0$,

$$\frac{1}{2} \frac{\partial}{\partial u} (V_i V^i) = -U_i \frac{\delta V^i}{\delta v}. \quad (195)$$

The normals and curvatures of a curve are defined mathematically by the Frenet-Serret formulae, which have the form 1-(55) for a timelike curve. For a spacelike curve, certain changes in sign have to be made, but we shall keep 1-(55a) unaltered:

$$\frac{\delta A^i}{\delta s} = b B^i, \quad b \geq 0, \quad (196)$$

where b is the first curvature and B^i the unit first normal ¹. Then (195)

¹ For simplicity we shall suppose that $\delta A^i/\delta s$ is spacelike or timelike. It might be null, in which case we would say that the curve has zero first curvature but a well-defined null first normal, the components of which could be measured chronometrically.

may be written

$$\frac{1}{2} \frac{\partial}{\partial u} (V_i V^i) = -b U_i B^i. \quad (197)$$

Let us consider two adjacent geodesics with parameters v and $v + \eta_0$, so that η_0 is their normal separation at C . Let η be their normal separation for any value of u ; then

$$V_i V^i = \eta^2 / \eta_0^2, \quad (198)$$

and (197) gives (on C)

$$\frac{1}{\eta_0} \frac{d\eta}{du} = -b U_i B^i. \quad (199)$$

If we now take a Fermi tetrad on C and resolve vectors on it, we get

$$\frac{1}{\eta_0} \frac{d\eta}{du} = -b B_{(a)} U^{(a)}. \quad (200)$$

Here all quantities have been given physical meanings except b and $B_{(a)}$, and the quantities $U^{(a)}$ are arbitrary except for

$$U_{(a)} U^{(a)} = -1. \quad (201)$$

Hence we have a physical meaning for the vector $bB_{(a)}$, and since

$$B_{(a)} B^{(a)} = \pm 1, \quad (202)$$

we can get b and $B_{(a)}$.

We have thus attached physical meanings to the first curvature and normal of a spacelike curve, and by 'physical meaning' we understand that these quantities can be measured chronometrically.

It is now easy to test a spacelike curve to see if it is a geodesic. All we have to do is to measure the left hand side of (199): if it is zero for all timelike vectors U^i , then C is a geodesic. It is very a simple idea really: in the case of a geodesic C , the orthogonal geodesics, starting from C with Fermi-transported (i.e. parallel transported) directions, do not open out or close in.

§ 13. THE PHYSICAL MEANING OF ABSOLUTE DIFFERENTIATION AND THE SYSTEMATIC MEASUREMENT OF GRAVITATIONAL FIELDS

Let C be any timelike or spacelike curve and $\lambda_{(a)}^i$ an orthonormal tetrad which undergoes Fermi-Walker transport along C , so that, by

I-(72) and (187),

$$\frac{\delta \lambda_{(a)}^i}{\delta s} = -\varepsilon \lambda_{(a)}^j \left(A^i \frac{\delta A_j}{\delta s} - A_j \frac{\delta A^i}{\delta s} \right), \quad (203)$$

where A^i is the unit tangent and ε is the indicator of C . Then any vector V^i defined along C may be resolved into invariant components,

$$V^i = V^{(a)} \lambda_{(a)}^i, \quad (204)$$

and the absolute derivative is

$$\frac{\delta V^i}{\delta s} = \frac{dV^{(a)}}{ds} \lambda_{(a)}^i - \varepsilon V^{(a)} \lambda_{(a)}^j \left(A^i \frac{\delta A_j}{\delta s} - A_j \frac{\delta A^i}{\delta s} \right). \quad (205)$$

The quantities on the right are physically measurable, and so the absolute derivative $\delta V^i/\delta s$ is physically measurable. The same holds of course for the absolute derivative with respect to any other parameter (it need not be s).

As has been remarked earlier, the ordinary measurements of 'gravitational fields' by means of a pendulum or, equivalently, the tension in a plumb line, are not in fact measurements of the gravitational field at all — they are measurements of the absolute acceleration of the observer. In § 11, where we put into relativistic form the experiment of weighing a body in the cellar against a similar body in the attic, we got nearer to the heart of the matter, i.e. the Riemann tensor, but the method was rather special and yielded only some of the components. Now that we have given a physical meaning to absolute derivatives, we are in a position to give a formula from which all the components can be obtained. No attempt is made here to describe the sort of apparatus which would be suitable for this type of relativistically valid geodetic survey.

The method is very general. We take a congruence of curves, preferably timelike, so that the curves may be regarded as the histories of a cloud of particles. There is no necessity that they should be free. We take four parameters $y_{(a)}$, with the first three $y_{(\alpha)}$ constant along each of the curves; $y_{(4)}$ is a parameter along each curve — it might be time, but need not be. The congruence is then described by equations of the form

$$x^i = x^i(y), \quad (206)$$

and we write

$$X_{(a)}^i = \frac{\partial x^i}{\partial y_{(a)}} \quad (207)$$

for the partial derivatives, these being four contravariant vector fields. By 1-(95) we have

$$\begin{aligned} R_{(abcd)} &= R_{ijkl} X_{(a)}^i X_{(b)}^j X_{(c)}^k X_{(d)}^l \\ &= g_{ij} X_{(a)}^i \left(\frac{\partial^2 X_{(b)}^j}{\partial y_{(c)} \partial y_{(d)}} - \frac{\partial^2 X_{(b)}^j}{\partial y_{(d)} \partial y_{(c)}} \right). \end{aligned} \quad (208)$$

This is the required formulae yielding the invariant components of the Riemann tensor in terms of absolute derivatives. One would for convenience choose the parameters $y_{(a)}$ so that, at the event under consideration, the vectors (207) form an orthonormal tetrad.

CHAPTER IV

THE MATERIAL CONTINUUM

§ 1. A STATISTICAL MODEL

Although we must never confound a mathematical model of nature with nature itself (cf. the philosophical remarks of III-§ 1), it certainly appears that the modern physicist finds a more realistic representation of matter in an assembly of particles than in a continuum. Accordingly this Chapter begins with a brief study of the statistics of an assembly of particles. But, since the general theory of relativity is essentially a field theory, we shall abandon the discrete model of matter as soon as it has served as a background against which the theory of a continuous medium may be constructed.

In Riemannian space-time we picture an assembly of *particles*, this term including

- (i) material particles,
- (ii) photons,
- (iii) internal impulses.

A material particle has a timelike world-line with 4-velocity v^i ($v_i v^i = -1$) and 4-momentum p^i ($= mv^i$), so that the square of the mass is $m^2 = -p_i p^i$. A photon has a null world-line with p^i tangent to it, but v^i does not exist and $m = 0$. A photon may be regarded as the limit of a material particle, with m tending to zero and v^i to infinity in such a way that mv^i tends to the finite limit p^i . An internal impulse is either repulsive or attractive. A repulsive impulse is mechanically the same as a photon, but an attractive impulse has its 4-momentum pointing into the past instead of into the future. It has in fact a negative energy, whereas all the other particles have positive energies. These internal impulses are of course highly hypothetical. They are introduced in order make the model applicable to a solid body which resists pressure and tension ¹.

¹ Cf. SYNGE [1956a, p. 210].

The various particles may collide with one another, i.e. their world-lines may intersect. At a collision there are abrupt changes in 4-momentum. In the present section it is unnecessary to assume (as we shall in § 2) that the world-lines are geodesics between collisions, nor is it necessary to assume any laws of conservation at a collision. Certain statistical quantities can be defined without such assumptions.

To discuss the statistics of an assembly consisting of a great number of particles, at any event O under consideration we introduce an orthonormal tetrad $\lambda_{(a)}^i$ with $\lambda_{(4)}^i$ timelike, and we resolve the 4-momentum p^i of any particle at O into invariant components [cf. I-(54)]

$$p_{(a)} = p_i \lambda_{(a)}^i. \quad (1)$$

Raising the Lorentz index, we may regard $p^{(a)}$ as rectangular coordinates in a flat Minkowskian 4-momentum space. In that space we have pseudospheres with equations

$$p_{(a)} p^{(a)} = \text{const.}, \quad (2)$$

and a null cone with equation

$$p_{(a)} p^{(a)} = 0. \quad (3)$$

There is an invariant element of 4-volume

$$dp = dp^{(1)} dp^{(2)} dp^{(3)} dp^{(4)}. \quad (4)$$

On a pseudosphere there is an invariant element of 3-volume and on the null cone there is an invariant element of 2-content ¹.

To deal competently with the statistics of photons and internal impulses we need to integrate over the null cone, and, if the material particles are quantized so that their masses take only certain discrete values, we need to integrate over pseudospheres ². However these 3-fold integrations introduce certain formal complexities which tend to hide what is important for our present purposes, and so we shall blur the picture a little. We shall give the photons and internal impulses timelike world-lines and very small masses, and we shall suppose the material particles unquantized. Further, to get a rapid view of essentials, we shall omit attractive internal impulses. What we have before us then is in fact an assembly of material particles with masses

¹ Cf. SYNGE [1956a, p. 430].

² Cf. SYNGE [1957c].

which can have any positive values, so that the representative points in 4-momentum space form a cloud inside the future sheet of the null cone (3). Should we wish to get back to the original picture, we would reinstate the attractive impulses and proceed to certain limits, pushing certain 4-momenta on to the null cone and perhaps others on to quantized pseudospheres. But with such processes we shall not be concerned.

With a chosen event O we associate some 4-momentum p^i , so that we have two points, the point O in space-time and a point in 4-momentum space. At O we take a *polarized target*¹ dS ; this is a 3-element with unit normal n^i , the sense of n^i polarizing dS . At the point p^i we take a 4-cell dp , and we are interested in the number of particles with world-lines cutting dS and with 4-momenta in dp . However, it is better to consider the *polarized number* of particles, this being the number prefixed with a $+$ or $-$ sign according as the particles make a positive or negative transit across dS , i.e. pass through in the sense of n^i or the opposite sense. The sense of transit is that of the 4-velocity v^i ($= p^i/m$), and it is easy to see that we get a positive or negative transit according as

$$\frac{\varepsilon(n)v_i n^i}{|v_j n^j|} = +1 \text{ or } -1. \quad (5)$$

Thus, to obtain the polarized number from the number itself, we have merely to multiply by this *polarization factor*.

We have now before us a thin tube of world-lines (Fig. 1), dS being an oblique section. If dS_0 is the normal section, we have the projection formula

$$dS_0 = dS |v_j n^j|. \quad (6)$$

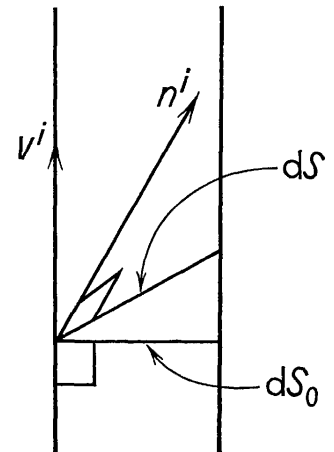


Fig. 1 – Thin timelike world tube with normal section dS_0 and oblique section dS

¹ The argument covers both spacelike and timelike targets, i.e. targets for which the normal n^i is timelike and spacelike, respectively. To polarize a spacelike target we may choose n^i pointing either into the future or into the past; to polarize a timelike target, we may choose either of the two opposed spacelike normals. It is useful to note that all vectors lying in a spacelike target are spacelike, whereas in the case of a timelike target some of the vectors lying in it are timelike and some are spacelike. We recall that all vectors orthogonal to a timelike vector are spacelike, but that a spacelike vector has spacelike, null and timelike vectors orthogonal to it.

The number of particles involved may be written

$$\nu(x, p)dS_0dp, \quad (7)$$

where $\nu(x, p)$ is a *distribution function*, obviously invariant from the construction. Substituting for dS_0 from (6) and introducing the polarization factor (5), we see that *the polarized number of particles in the class $(x, dS; p, dp)$ is*

$$\nu(x, p)\varepsilon(n)v_in^idSdp. \quad (8)$$

So far we have considered only one thin tube. Now, keeping the polarized target fixed, we take all particles into consideration, and study certain fluxes across the polarized target, such as the flux of number and the flux of 4-momentum. Since the same general argument applies to them all, we shall speak of a q -flux, where q is some quantity associated with a particle. It may be a scalar, vector, or tensor, but we suppress the indices for the present. The q -flux across the polarized target dS is defined as

$$q\text{-flux} = \sum_+ q - \sum_- q, \quad (9)$$

where \sum_+ is the sum of all the q -values for particles making positive transits and \sum_- the sum of all the q -values for particles making negative transits. Understanding that q has the same value for all particles in a class $(x, dS; p, dp)$ (as of course it will if q is a function of 4-momentum only), we evaluate the q -flux from (9) by multiplying q by the polarized number (8) and integrating over 4-momentum space. The minus sign in (9) is automatically taken care of by the polarization of the number, and we get

$$q\text{-flux} = \varepsilon(n)Q_in^idS, \quad (10)$$

where

$$Q_i = \int \nu(x, p)qv_idp = \int \nu(x, p)m^{-1}qp_idp. \quad (11)$$

The point of writing the q -flux in this form is that the target is separated from the statistical quantity Q^i which is independent of the target's size and orientation, although of course it depends on its position. Note that Q_i is a vector only if q is a scalar. If we indicate the tensor character of q by writing $q_{...}$, then the tensor character of Q_i is indicated by $Q_{...i}$.

If we give to q in turn the values

$$q = 1, \quad p_j, \quad p_j p_k, \quad p_j p_k p_m, \quad \dots \quad (12)$$

we get the following *moments* of the distribution function $\nu(x, p)$:

$$q = 1; \quad Q_i = \int \nu(x, p) m^{-1} p_i dp, \quad (13)$$

$$q = p_j; \quad Q_{ij} = \int \nu(x, p) m^{-1} p_i p_j dp, \quad (14)$$

$$q = p_j p_k; \quad Q_{ijk} = \int \nu(x, p) m^{-1} p_i p_j p_k dp, \quad (15)$$

$$q = p_j p_k p_m; \quad Q_{ijkm} = \int \nu(x, p) m^{-1} p_i p_j p_k p_m dp, \quad (16)$$

and so on. The corresponding fluxes across a polarized target dS are

$$q = 1; \quad q\text{-flux} = \varepsilon(n) Q_i n^i dS, \quad (17)$$

$$q = p_j; \quad q\text{-flux} = \varepsilon(n) Q_{ij} n^j dS, \quad (18)$$

$$q = p_j p_k; \quad q\text{-flux} = \varepsilon(n) Q_{ijk} n^k dS, \quad (19)$$

$$q = p_j p_k p_m; \quad q\text{-flux} = \varepsilon(n) Q_{ijkm} n^m dS. \quad (20)$$

To complete the mathematical list of moments, we should add

$$Q = \int \nu(x, p) dp, \quad (21)$$

but this appears to have no physical meaning in our model.

We might of course have multiplied the q 's in (12) by any powers of the mass m . But this is inadvisable, since we would like to have moments which remain finite in the limit $m \rightarrow 0$, i.e. when we pass from material particles to photons or internal impulses.

Of the moments listed above, two are of particular interest. These are (13) and (14), and to them we give special names and use a different notation. Writing

$$N_i = \int \nu(x, p) m^{-1} p_i dp = \int \nu(x, p) v_i dp, \quad (22)$$

we call this the *numerical vector*, and (putting $q = 1$ in (9)) we see that

$$\varepsilon(n) N_i n^i dS \quad (23)$$

is the *numerical flux* across the polarized target dS , i.e. the number of particles with positive transits less the number with negative transits. Turning to (14), we write

$$T_{ij} = \int \nu(x, p) m^{-1} p_i p_j dp = \int \nu(x, p) v_i p_j dp, \quad (24)$$

and call this the *energy tensor*; we note its symmetry,

$$T_{ij} = T_{ji}, \quad (25)$$

which is very important. Then, by (18), the flux of covariant 4-momentum p_i across a polarized target dS is

$$\varepsilon(n) T_{ij} n^j dS; \quad (26)$$

putting $q = p_i$ in (9), we see that this represents the total covariant 4-momentum of particles making positive transits across dS less the total covariant 4-momentum of those making negative transits.

Since we have worked entirely in the small, general relativity has played no part — we might have been in flat space-time. If we use the members $\lambda_{(\alpha)}^i$ of the orthonormal tetrad as space-axes and $\lambda_{(4)}^i$ as time-axis, we can give useful verbal descriptions of the invariant components of N_i and T_{ij} by taking the vector n^i along each of the vectors of the tetrad in turn. Thus

$$N^{(\alpha)} = N_{(\alpha)} = \text{polarized number of particles crossing} \\ \text{unit area per unit time}, \quad (27)$$

$$N^{(4)} = -N_{(4)} = \text{number of particles per unit volume.}$$

Making use of the symmetry of T_{ij} and defining stress by flux of 3-momentum with the usual convention which makes tension positive and pressure negative, we have

$$\begin{aligned} -T^{(\alpha\beta)} &= -T_{(\alpha\beta)} = \text{stress matrix,} \\ T^{(\alpha 4)} &= T^{(4\alpha)} = -T_{(\alpha 4)} = -T_{(4\alpha)} \\ &= \text{polarized flux of energy per unit area} \\ &\quad \text{per unit time} \\ &= \text{density of 3-momentum,} \\ T^{(44)} &= T_{(44)} = \text{density of energy.} \end{aligned} \quad (28)$$

We raise and lower Lorentz indices as in I-(54); values are not altered by raising or lowering 1, 2, 3, but there is a change of sign each time we raise or lower 4.

§ 2. CONSERVATION LAWS IN THE STATISTICAL MODEL

So far our statistical model has been developed without the geodesic hypothesis and without any law of conservation at collisions. We now introduce the *geodesic hypothesis*: the world-line of each particle is a geodesic between collisions, with p^i undergoing parallel transport. As regards conservation, the conservation of 4-momentum seems to be one of the most fundamental laws of physics, and we shall assume that at each collision the total 4-momentum is conserved. Since each collision is a single event, this does not involve adding vectors attached to different events.

The question of the conservation of number is more dubious. In the kinetic theory of gases we would assume conservation of number, if we were prepared to leave all photons out of account. But in view of the annihilation and creation of particles and in view of the presence of radiation in the form of photons (not to speak of the hypothetical internal impulses), we hesitate to place conservation of number on a parity with conservation of 4-momentum.

We proceed then with the geodesic hypothesis and the law of conservation of 4-momentum. Let S be a closed 3-space in space-time (Fig. 2). We polarize it by the outward unit normal n^i . Then, as in (26), the flux of 4-momentum across an element dS is

$$\text{flux of 4-momentum} = \varepsilon(n) T_{ij} n^j dS. \quad (29)$$

We cannot find the total flux of 4-momentum across S by integrating this, because we are in curved space-time and we must not add vectors at different points. So we introduce a vector field W^i , chosen arbitrarily at an event O inside S , and defined throughout R (the interior of S) by parallel transport along the geodesic emanating from O . It follows from this construction that

$$W_{i;j} = 0 \text{ at } O. \quad (30)$$

We take components on W^i , and in view of (29) and (9) we have the following two equal expressions:

total flux of the W -component of 4-momentum across S

$$= \int \varepsilon(n) T^{ij} W_i n_j dS = \sum_+ W_i p^i - \sum_- W_i p^i, \quad (31)$$

where \sum_+ refers to particles leaving R (positive transit) and \sum_- to particles entering R (negative transit).

We take S small, with linear dimensions of order σ , and follow the usual convenient but confusing practice of regarding the number of particles involved as very great in spite of the smallness of σ . We have to estimate the order of magnitude of the expression

$$F = \sum_{+} W_i p^i - \sum_{-} W_i p^i, \quad (32)$$

which occurs in (31). We do this in two steps. First, we allow all particles entering R to pursue their geodesic world-lines without collisions, passing unbroken through the actual event of collision, as at C in Fig. 2a. Secondly, we study the effects of the collisions, one by one.

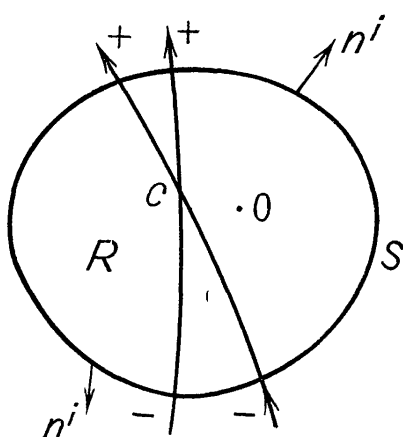


Fig. 2a – No collisions

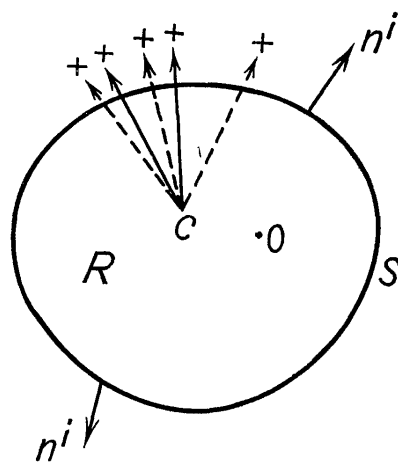


Fig. 2b – Collision

As we follow a particle, we have

$$\frac{d}{ds} (W_i p^i) = W_{i|j} p^i \frac{dx^j}{ds}, \quad (33)$$

By (30) $W_{i|j}$ is of the order σ in R and the range of s is also of this order. Therefore, in the absence of collisions, this single particle gives to F a contribution of order σ^2 . Now the number of particles is proportional to σ^3 , and so, in the absence of collisions,

$$F = O(\sigma^5). \quad (34)$$

Consider the effect on F of the collision at C in Fig. 2b. In (32) \sum_{-} refers to particles entering R , and the collision does not affect them. What the collision does is to change the contributions from the full lines in Figs. 2a and b into the contributions from the broken lines in Fig. 2b, these being the world-lines after the collision. If we denote the

former contribution by Σ and the latter by Σ' , we have to evaluate

$$\Sigma W_i p^i - \Sigma' W_i p^i, \quad (35)$$

this being the change in F produced by the collision. The expression (35) has to be evaluated at the events where the world-lines leave R , but the several elements are functions of position on the world-lines drawn from C to the events of departure from R . Now by the conservation of 4-momentum at C , (35) vanishes at C , and if we follow the quantities to the events of departure from R , using (33), we find in fact that (35) is of the order σ^2 . The total number of collision will be proportional to the 4-volume of R , i.e. proportional to σ^4 , and so we conclude that the effect of collisions is to change F only by the order σ^6 , which is insignificant in view of (34). Thus the effect of collisions is negligible, but this is due to conservation of 4-momentum, not to the paucity of collisions.

By (31), (32) and (34), we have

$$\int \varepsilon(n) T^{ij} W_i n_j dS = O(\sigma^5). \quad (36)$$

By Green's theorem as in I-(257), this may be written

$$\int T^{ij}_{|j} W_i d\tau + \int T^{ij} W_{i|j} d\tau = O(\sigma^5), \quad (37)$$

where the integrals are taken through R , $d\tau$ being an element of 4-volume. By (30) the second integral is of order σ^5 , and so, dividing by σ^4 , and going to the limit $\sigma \rightarrow 0$ with R collapsing on O , we get

$$T^{ij}_{|j} W_i = 0. \quad (38)$$

But W_i was chosen arbitrarily at O . We have then the following *differential equations of conservation of 4-momentum*:

$$T^{ij}_{|j} = 0. \quad (39)$$

It is most important to note that these are *differential* equations; we do not get an integrated law of conservation, and the absence of a simple integrated law has been one of the banes of general relativity, at least for those who expect to see Newtonian facts¹ reproduced in curved space-time.

As for the conservation of number, the argument is very simple. If we assume the conservation of number at each collision (we do not have to assume geodesic world-lines or conservation of 4-momentum), then the number of particles entering a domain R is equal to the num-

¹ Or those of special relativity. Cf. SYNGE [1956a, p. 311].

ber leaving R . Hence, if S bounds R as in Fig. 2, (23) gives

$$\int \varepsilon(n) N^i n_i dS = 0. \quad (40)$$

By Green's theorem this may be written

$$\int N^i_{|i} d\tau = 0, \quad (41)$$

and, shrinking R to a single event, we obtain the *differential equation of conservation of number*:

$$N^i_{|i} = 0. \quad (42)$$

But, as indicated earlier, this equation is to be regarded as less universally valid than (39).

Apart from conservation equations, we have in N^i [cf. (22)] and T_{ij} [cf. (24)] a vector and a tensor defined by our statistical model, and the question arises: How should we define the *mean velocity* of the assembly of particles? This question is of considerable importance in setting up the mechanics of a continuum if we are to base the structure of that theory on consideration of the statistical model, for the mean velocity V^i of a continuum is a fundamental concept.

Since velocity is a kinematical concept, we naturally seek a kinematical definition, and N^i is the only vector available. Thus we are led to define the *kinematical mean velocity* V^i_K as the unit vector having the direction of the numerical vector N^i . For a gas consisting of equal molecules this definition is reasonable, but if there are also present a considerable number of particles with masses very small compared with those of the molecules, it seems hardly right to define mean velocity on the basis of a mere counting. If we abandon the idea that mean velocity must be defined kinematically, we may define the *dynamical mean velocity* V^i_D as the unit vector in the timelike eigen-direction of the tensor T_{ij} , assuming such a timelike eigen-direction to exist, as it does under certain reasonable conditions¹. This means that V^i_D satisfies the equations

$$T_{ij} V^i_D = -\mu_D V_{Di}; \quad (43)$$

where $-\mu_D$ is an eigenvalue of T_{ij} .

Since a medium at rest is easier to think about than one in motion, it is natural to seek a *rest-frame* at each event in the history of the medium, this rest frame being an orthonormal tetrad with its fourth

¹ Cf. SYNGE [1956a, p. 292].

vector pointing in the direction of the mean velocity. Having got a rest-frame, we can proceed to talk about rest-energy and other rest-quantities. Great confusion of thought results from a tacit assumption that the mean velocity (and hence the rest-frame) is well defined when in fact it is not; it might be V_K^i or it might be V_D^i . In the former case the rest-energy is

$$\mu_K = T_{ij} V_K^i V_K^j, \quad (44)$$

and in the latter case it is

$$\mu_D = T_{ij} V_D^i V_D^j. \quad (45)$$

The question of the distinction between these two rival definitions of mean velocity would disappear if we could assert that the two mean velocities were the same, as indeed they seem to be for a gas in statistical (adiabatic) equilibrium. But there does not seem to be any justification for such an assumption in general, although it might be hard to find a physical example where the two vectors differed by any significant amount. For the sake of simplicity and clarity it is necessary to make a choice, and in this book the *dynamical mean velocity* V_D^i (an eigenvector of the energy tensor) will be accepted as the definition of mean velocity; if on any occasion it is desirable to deviate from this definition, the fact will be explicitly noted (cf. x-§ 1).

§ 3. KINEMATICS OF A CONTINUUM

Leaving the statistical model behind, we consider a continuum of *identifiable particles*, if we like to continue to use the word 'particle' — it now means something quite different from what it meant in the two preceding sections. In the present section the inertial properties of the continuum do not interest us. We study only its kinematics and that involves nothing but a field of 4-velocity V^i (it might be either of the mean velocities discussed in § 2). Our kinematics is merely the geometry of the *stream-lines*, these being world-lines having V^i for unit tangents. There are two methods — the Lagrangian and the Eulerian.

In the Lagrangian method, we introduce four parameters $y_{(\alpha)}$, the first three parameters $y_{(\alpha)}$ being constant along each stream-line. The equations of the congruence of stream-lines then appear in the form

$$x^i = x^i(y), \quad (46)$$

$y_{(4)}$ being a parameter (possibly the time) which varies along each stream-line, and the 4-velocity is

$$V^i = \theta \frac{\partial x^i}{\partial y_{(4)}}, \quad (47)$$

the scalar factor θ being chosen so that

$$V_i V^i = -1. \quad (48)$$

We shall not pursue the Lagrangian method further, because the Eulerian method is more convenient for tensorial treatment. In this method, we define the congruence of stream-lines by writing

$$V^i = V^i(x), \quad (49)$$

thus describing the 4-velocity as a function of position in space-time, the condition (48) being of course satisfied by these functions.

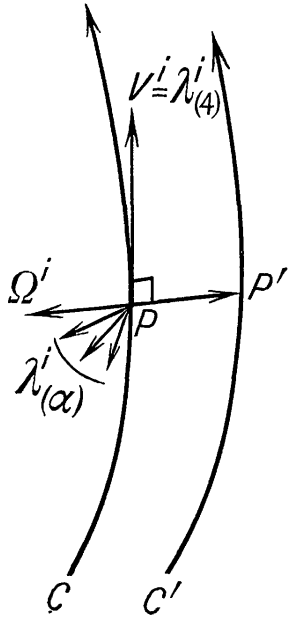


Fig. 3 – Kinematics of a continuum

The kinematics of the continuum deals with the relative behaviour of adjacent stream-lines, and may be discussed in various ways. We shall here use Fermi coordinates as in II-§ 10, taking a Fermi tetrad $\lambda_{(a)}^i$ on one of the stream-lines C , with $\lambda_{(4)}^i = V^i$. Although we are interested only in the immediate neighbourhood of C , this method has the advantage that we could go further if desired.

Fig. 3 shows the stream-line C and an adjacent stream-line C' , with the events P and P' in correspondence through the condition that the geodesic PP' is orthogonal to C at P . Then, in terms of the world-function, the Fermi coordinates of P' are

$$X_{(\alpha)} = -\Omega_i(PP')\lambda_{(\alpha)}^i, \quad (50)$$

and their rates of change with respect to time s on C are (with $D = d/ds$)

$$DX_{(\alpha)} = -\Omega_{ij}\lambda_{(\alpha)}^i V^j - \Omega_{ij'}\lambda_{(\alpha)}^i V^{j'}Ds', \quad (51)$$

where $V^{j'}$ is 4-velocity on C' and s' is time on C' ; in view of the orthogonality at P , no term arises from the differentiation of the Fermi vector. The calculation is very simple since we wish to retain only terms of the first order (O_1) in the Fermi coordinates. Remembering

the coincidence limits in II-(69), we find

$$\begin{aligned}\Omega_{ij}\lambda_{(\alpha)}^i V^j &= O_2, \\ \Omega_{ij'}\lambda_{(\alpha)}^i V^{j'} &= [\Omega_{ij'}V_{.k'}^{j'}]\lambda_{(\alpha)}^i \eta^k + O_2 \\ &= -\lambda_{(\alpha)}^i V_{ik}\eta^k + O_2, \\ Ds' &= 1 + O_1,\end{aligned}\tag{52}$$

where η^k is the infinitesimal vector PP' and the second subscript on V indicates a covariant derivative. Hence, to the first order,

$$DX_{(\alpha)} = V_{(\alpha\beta)}X^{(\beta)},\tag{53}$$

where

$$V_{(\alpha\beta)} = V_{ij}\lambda_{(\alpha)}^i\lambda_{(\beta)}^j,\tag{54}$$

these being the components of the covariant derivative V_{ij} on the Fermi triad.

Regarding the Fermi coordinates $X_{(\alpha)}$ as Cartesian, we recognize in (53) the equations of motion of a continuum undergoing linear deformation. Accordingly we shall speak of a symmetric *rate-of-strain matrix* given by

$$\sigma_{(\alpha\beta)} = \frac{1}{2}(V_{(\alpha\beta)} + V_{(\beta\alpha)}),\tag{55}$$

and a skew-symmetric *spin (or rotation) matrix* given by

$$\omega_{(\alpha\beta)} = \frac{1}{2}(V_{(\alpha\beta)} - V_{(\beta\alpha)}).\tag{56}$$

However, it is desirable to work with tensors rather than with invariants, and so we seek a symmetric tensor σ_{ij} and a skew-symmetric tensor ω_{ij} which have respectively the components (55) and (56) on the Fermi triad. But, since these components do not serve to define the tensors completely, we are at liberty to add other conditions, and for these conditions we shall make the choice which seems simplest, namely,

$$\sigma_{(a4)} = 0, \quad \omega_{(a4)} = 0;\tag{57}$$

these are equivalent to the tensor conditions

$$\sigma_{ij}V^j = 0, \quad \omega_{ij}V^j = 0.\tag{58}$$

To determine σ_{ij} from this condition and (55), which is the same as

$$\sigma_{km}\lambda_{(\alpha)}^m\lambda_{(\beta)}^k = \frac{1}{2}V_{km}(\lambda_{(\alpha)}^k\lambda_{(\beta)}^m + \lambda_{(\beta)}^k\lambda_{(\alpha)}^m),\tag{59}$$

we multiply (59) by $\lambda_i^{(\alpha)}\lambda_j^{(\beta)}$ and introduce the *projection operator* defined as

$$P_j^i = \lambda_{(\alpha)}^i \lambda_j^{(\alpha)} = \delta_j^i + V^i V_j. \quad (60)$$

To reduce the resulting expression, we use (58) and also, by (48),

$$V^i V_{ij} = 0. \quad (61)$$

In this way we obtain the following *rate-of-strain tensor*:

$$\sigma_{ij} = \frac{1}{2} V_{km} (P_i^k P_j^m + P_j^k P_i^m) = \frac{1}{2} (V_{ij} + V_{ji} + V_{ik} V^k V_j + V_{jk} V^k V_i). \quad (62)$$

This tensor has of course, not 10 independent components as might appear, but only 6 in view of (58), or, better, in view of the fact that σ_{ij} is uniquely determined by the six components (55) with (57).

The same type of argument, based on (56) and (58), leads to the following *spin tensor*:

$$\omega_{ij} = \frac{1}{2} V_{km} (P_i^k P_j^m - P_j^k P_i^m) = \frac{1}{2} (V_{ij} - V_{ji} + V_{ik} V^k V_j - V_{jk} V^k V_i). \quad (63)$$

This tensor has actually only three independent components, for when we resolve it on the Fermi tetrad we get only the independent components $\omega_{(23)}$, $\omega_{(31)}$, $\omega_{(12)}$.

The spin tensor is closely connected with the *spin vector* defined by

$$\omega^i = \frac{1}{2} \eta^{ijkm} V_j V_{km}, \quad (64)$$

the η -term being the permutation tensor as in I-(114). The components of this vector on the Fermi tetrad are

$$\omega^{(a)} = \frac{1}{2} \zeta \varepsilon_{abcd} V_{(b)} V_{(cd)}, \quad \zeta^{-1} = (-g)^{\frac{1}{2}} \det \lambda_{(a)}^i = \pm 1. \quad (65)$$

Now $V_{(3)} = 0$, $V_{(4)} = -1$, and so

$$\omega^{(\alpha)} = \frac{1}{2} \zeta \varepsilon_{\alpha\gamma\delta} V_{(\gamma\delta)} = \frac{1}{2} \zeta \varepsilon_{\alpha\gamma\delta} \omega_{(\gamma\delta)}, \quad \omega^{(4)} = 0, \quad (66)$$

so that

$$\omega^{(1)} = \zeta \omega_{(23)}, \quad \omega^{(2)} = \zeta \omega_{(31)}, \quad \omega^{(3)} = \zeta \omega_{(12)}; \quad (67)$$

the spin tensor and the spin vector are essentially the same thing in different forms.

Let v be the 3-volume of the normal section of a thin tube of streamlines. Then, as may be shown, the *expansion* of the tube is

$$\frac{1}{v} \frac{dv}{ds} = g^{ij} \sigma_{ij} = V^i_{;i}, \quad (68)$$

and so the condition for a motion without expansion (incompressible) is

$$g^{ij}\sigma_{ij} = 0 \text{ or } V^i_{;i} = 0. \quad (69)$$

A motion is *rigid* (in the sense of Born, cf. III-§ 5) if, for all stream-lines adjacent to C , the vector PP' of Fig. 3 retains a constant magnitude or equivalently if $X_{(\alpha)}X^{(\alpha)} = \text{const}$. It is clear from (53) that necessary and sufficient conditions for a rigid motion are

$$\sigma_{(\alpha\beta)} = 0 \text{ or } \sigma_{ij} = 0, \quad (70)$$

these conditions being equivalent to one another.¹

A motion may be called *irrotational* if

$$\omega_{(\alpha\beta)} = 0 \text{ or } \omega_{ij} = 0, \quad (71)$$

these conditions being equivalent to one another. By (67) these conditions are also equivalent to $\omega^i = 0$. But in writing out these last conditions, we may replace the permutation tensor in (64) by the numerical permutation symbol, and we may also replace the covariant derivative by a partial derivative. We then recognize the well known integrability conditions for the total differential equation

$$V_i dx^i = 0. \quad (72)$$

We conclude that, *in an irrotational motion, the stream-lines form a normal congruence*; in other words, there exists a family of 3-spaces to which the stream-lines are orthogonal. There has been so much vague thought about *rotation* in relativity that it is well to emphasize how simple the matter is: in irrotational motion the element does not rotate relative to axes carried by Fermi transport along a world-line of the continuous medium.

§ 4. THE ENERGY TENSOR OF A CONTINUUM

Following the suggestion of the statistical model (§§ 1, 2), we assign to a material continuum a *symmetric energy tensor*, for which we can use the covariant form T_{ij} , the contravariant form T^{ij} , or the mixed form T^i_j ; in view of the symmetry it is unnecessary to distinguish between $T^i_{;j}$ and $T_{;j}^i$ and we write them both $T^i_{;j}$.

The energy tensor plays two important roles. First, it embodies the

¹ Cf. ROSEN [1947], SALZMAN and TAUB [1954]. For other work on rigid motions, see RAYNER [1959]. The problem of rigid motions in general relativity is more difficult than was indicated in SYNGE [1956a, p. 36] on account of the parabolic nature of the system of equations.

mechanical properties of the matter, such as stress and density. Secondly, it is the definitive quantity in the determination of gravitational fields, in much the same way as density is the definitive quantity in Newtonian gravitation. The fact that the energy tensor plays this dual role, inertial and gravitational, is sometimes referred to as the *equivalence of inertial and gravitational mass*. Deferring to the next section the gravitational field equations, we proceed with the inertial aspects of the energy tensor.

We borrow from the statistical model the interpretation of the energy tensor in terms of fluxes, and as in (26) we make the following statement:

$$\begin{aligned} &\text{flux of 4-momentum across a 3-target } dS \text{ polarized by a} \\ &\text{unit normal vector } n^i = \varepsilon(n) T^{ij} n_j dS. \end{aligned} \quad (73)$$

For the meanings of the components on an orthonormal tetrad, see (28); but whether the words used there convey precise physical meanings depends to no little extent on the experience of the reader and his imagination.

We also take from the statistical model the *conservation equation*

$$T^{ij}_{|j} = 0. \quad (74)$$

It is important to note that if the metric tensor g_{ij} is given, then we have here four partial differential equations satisfied by the ten components of T^{ij} ; but if g_{ij} is not given, then these are differential equations for g_{ij} as well as T^{ij} , since the first derivatives of the former occur.

As indicated at the end of § 2, we define the *4-velocity* V^i of the continuum as the timelike eigenvector of T_{ij} , so that

$$T_{ij} V^j = -\mu V_i, \quad (75)$$

the invariant μ being called the *proper density of energy or of mass*¹; but for brevity we shall omit the word *proper* unless there is risk of confusion with the density of (28), which becomes the proper density if we take the reference vector $\lambda^i_{(4)}$ to be V^i . We have

$$V_i V^i = -1, \quad V_i V^i_{|j} = 0, \quad (76)$$

¹ In the chronometric approach used in this book, the speed of light is automatically unity. Thus mass and proper energy are the same thing and Einstein's famous equation $E = mc^2$ reads simply $E = m$.

and

$$\mu = T_{ij}V^iV^j. \quad (77)$$

The minus sign occurs in (75) because, in the statistical model, the 4-momenta of material particles and photons point into the future.

The symmetric tensor S_{ij} defined by

$$T_{ij} = \mu V_i V_j - S_{ij} \quad (78)$$

is called the *stress tensor*, this being in agreement with (28); by (75) we have

$$S_{ij}V^j = 0, \quad (79)$$

so that S_{ij} has only six independent components.

In (78) we have the general expression for the energy tensor in terms of density, 4-velocity and stress. The simplest of all continua is the *incoherent fluid* or *dust cloud*, defined by the condition $S_{ij} = 0$, so that

$$T_{ij} = \mu V_i V_j. \quad (80)$$

To investigate the stream-lines of an incoherent fluid, we substitute (80) in (74) and obtain

$$(\mu V^j)_{|j} V^i + \mu V^i_{|j} V^j = 0. \quad (81)$$

Multiply by V_i and use (76): this gives

$$(\mu V^j)_{|j} = 0, \quad (82)$$

and so by (81)

$$DV^i = V^i_{|j} V^j = 0, \quad (83)$$

where $D = \delta/\delta s$, the operator of absolute differentiation along the stream-line. Therefore *in an incoherent fluid, the stream-lines are geodesics*, a rather remarkable result.

Next in order of simplicity comes the *perfect fluid*, defined by the condition that T_{ij} has three equal eigenvalues corresponding to space-like eigenvectors. Denoting the common eigenvalue by p (*pressure*), we have

$$T_{ij} = \mu V_i V_j + p(V_i V_j + g_{ij}) = (\mu + p)V_i V_j + p g_{ij}. \quad (84)$$

To investigate the motion of a perfect fluid, we substitute from (84) in (74) and obtain

$$[(\mu + p)V^j]_{|j} V^i + (\mu + p)V^i_{|j} V^j + p_{|i} = 0. \quad (85)$$

On multiplying by V^i and using (76), we get an equation which can be

written in either of the following forms:

$$[(\mu + p)V^j]_{|j} = p_{|i}V^i, \quad (86)$$

$$pV^i_{|j} = -(\mu V^j)_{|j}. \quad (87)$$

Substitution of (86) in (85) gives

$$(\mu + p)DV^i = -p_{|j}(V^iV^j + g^{ij}), \quad (88)$$

which expresses the absolute acceleration of a stream-line in terms of the pressure-gradient. If we introduce an orthonormal tetrad $\lambda^i_{(a)}$ with $\lambda^i_{(4)} = V^i$ and denote in an obvious way the invariant components of $p_{|i}$ and DV^i on this tetrad, we have

$$[(\mu + p)V^j]_{|j} = p_{(4)}, \quad (89)$$

$$(\mu + p)(DV)_{(\alpha)} = -p_{(\alpha)}. \quad (90)$$

We recognize in this last equation a modification of the Newtonian equation of hydrodynamics: density \times acceleration = — pressure-gradient. In nature, p/μ is very small [cf. (98)].

A gas composed of identical molecules of mass m , in statistical adiabatic equilibrium, is a particular case of a perfect fluid. For it the kinematical and dynamical velocities (cf. § 2) coincide, and the energy tensor is ¹

$$T_{ij} = mNG(m\xi)V_iV_j + g_{ij}N/\xi, \quad (91)$$

where ξ is the reciprocal temperature, $G = K_3/K_2$ (a ratio of Bessel functions), and N is the magnitude of the numerical vector, so that

$$N^i = NV^i. \quad (92)$$

We have not only the equation of conservation (74), but also the conservation of number, so that, as in (42),

$$N^i_{|i} = 0. \quad (93)$$

Comparison of (80) and (84) shows that a perfect fluid degenerates into an incoherent fluid when p tends to zero, and this suggests that it may be legitimate to treat a perfect fluid at low pressure as if it

¹ Cf. SYNGE [1957c, p. 36].

were incoherent. But what does 'low pressure' mean? In ordinary units, energy-density and pressure both have the dimensions $[ML^{-1}T^{-2}]$; by making a chronometric definition of distance, we have set up the dimensional equation $[L] = [T]$, and so we may write

$$[\mu] = [p] = [MT^{-3}]. \quad (94)$$

In the next section we shall see that $[M] = [T]$ by virtue of the field equations, so that μ and p are expressible in sec^{-2} . Since they are dimensional, there is no sense in talking about μ or p being large or small. But we can speak with meaning of the magnitude of the dimensionless ratio p/μ , and it is of interest to investigate this magnitude in the case of the earth, considered as a perfect fluid with μ constant, and without rotation.

The first curvature of the world-line of any particle fixed in the earth is the local 'acceleration due to gravity' g [cf. III-(133)], and (90) may be written

$$\frac{1}{\mu + p} \frac{dp}{dr} = -g, \quad (95)$$

with g evaluated at a distance r from the centre of the earth. Let us accept from Newtonian theory

$$g = g_1 r/r_1, \quad p_1 = 0, \quad (96)$$

the subscript 1 referring to the earth's surface. Then

$$\log(1 + p_0/\mu) = \frac{1}{2}g_1 r_1 \quad (97)$$

where p_0 is the pressure at the earth's centre. The quantity $g_1 r_1$ is (chronometrically) dimensionless, and we have

$$\begin{aligned} g_1 &= 3.263 \times 10^{-8} \text{ sec}^{-1}, \\ r_1 &= 2.125 \times 10^{-2} \text{ sec}, \\ p_0/\mu &= \frac{1}{2}g_1 r_1 = 3.464 \times 10^{-10}. \end{aligned} \quad (98)$$

This ratio is so fantastically small that one's first reaction is to drop p from (84) in the case of any reasonable fluid and treat it as an incoherent fluid. But this would be disastrous. In the absence of all pressure, a ship launched on the ocean would pursue a geodesic, i.e. it would sink to the bottom, and indeed the ocean would collapse [cf. (83)]. The explanation of this is as follows. If q is a very small

dimensionless quantity, we may properly neglect it in $(1 + q)$. But the gradient of q is not dimensionless, and it may be neglected only in comparison with quantities of the same dimensions. It would be ridiculous to neglect the left hand side of (95) in comparison with the right! ¹

We now turn to the general energy tensor (78) and substitute it in the conservation equation (74), obtaining

$$(\mu V^j)_{|j} V_i + \mu V_{i|j} V^j = S^{ij}_{|j}. \quad (99)$$

Multiplication by V^i gives

$$(\mu V^j)_{|j} = - V_i S^{ij}_{|j}, \quad (100)$$

and so we get from (99)

$$\mu D V_i = (g_{ij} + V_i V_j) S^{jk}_{|k}. \quad (101)$$

Eq. (100) may also be written in the form

$$D\mu = - \mu V^j_{|j} - V_i S^{ij}_{|j}. \quad (102)$$

By (79) we have

$$S^{jk} V_j = 0, \quad S^{jk}_{|k} V_j + S^{jk} V_{j|k} = 0, \quad (103)$$

and so, in terms of the rate-of-strain tensor σ_{ij} of § 3,

$$V_j S^{jk}_{|k} = - S^{jk} \sigma_{jk}. \quad (104)$$

We can now write (101) and (102) in the form

$$\mu D V^i = S^{ij}_{|j} - V^i S^{jk} \sigma_{jk}, \quad (105)$$

$$D\mu = - \mu \sigma^k_k + S^{jk} \sigma_{jk}, \quad (106)$$

the first equation giving the absolute acceleration (or first curvature) of a stream-line, and second giving the rate of change of density along it. These may be regarded as the *equations of motion of a general continuous medium*, although of course the number of unknowns exceeds the number of equations.

The point of writing the equations in this last form is to bring the rate-of-strain tensor σ_{ij} into play. Since all actual substances are imperfectly elastic or viscous, as long as σ_{ij} does not vanish heat will be generated and radiated off from the surface of any finite body. This

¹ Cf. remarks on smallness in II-§ 3.

suggests that the planets tend to states in which σ_{ij} vanishes (i.e. to rigid motions) and for such states we have

$$\mu DV^i = S^i_{|j}, \quad \sigma_{ij} = 0, \quad (107)$$

with μ constant along each stream-line.

§ 5. THE FIELD EQUATIONS AND THE NEWTONIAN COMPARISON

So far we have assumed that space-time is a Riemannian 4-space and have, in general terms, attributed its curvature to gravitation. But we have set up no equations by which the curvature of space-time is made to depend on the distribution of matter. This gap is now filled by writing down Einstein's field equations, which read

$$G_{ij} - \Lambda g_{ij} = -\kappa T_{ij}. \quad (108)$$

In these equations G_{ij} is the Einstein tensor, g_{ij} the metric tensor, T_{ij} the energy tensor, and Λ and κ are two universal constants. The field equations can of course be written also in contravariant and mixed forms.

Since we have already committed ourselves to the conservation equation (74), we must assure ourselves that it is consistent with (108). This consistency is at once established if we recall the identity I-(111) satisfied by the Einstein tensor, for we get from (108)

$$\kappa T^i_{|j} = \Lambda g^i_{|j} - G^i_{|j} = 0. \quad (109)$$

Let us now consider (108) from the standpoint of physical dimensions. In general relativity only invariants have definite physical dimensions. The physical dimensions of the components of a tensor depend on those of the coordinates employed, but it is clear that in each component of a tensor equation all the terms must have the same dimensions. This relieves us of the trouble of constructing invariants. It is a useful plan to think of coordinates which are times (seconds). Then, since the element ds is itself a time, g_{ij} is dimensionless and G_{ij} has the dimensions of sec^{-2} , since it involves two differentiations with respect to the coordinates. We recognize then that Λ has the dimensions sec^{-2} . But we cannot assign dimensions in seconds to κ because T_{ij} involves mass, and so far that is a separate quantity dimensionally. We now force a dimensionality on mass (or equivalently energy) by insisting that κ shall be a pure number. It would seem

simplest to choose κ equal to unity, but if we do this an awkward factor will appear later. To forestall this, we choose

$$\kappa = 8\pi; \quad (110)$$

but as κ is easier to write, we shall continue to use that symbol for this pure number.

It is clear from (108) that now density has the same dimensions as Λ , that is, sec^{-2} . Hence *mass or energy is a time, to be measured in seconds*. But we shall not be able to express grammes in seconds until we have compared the relativistic gravitational field with physical reality.

The constant Λ is called the *cosmological constant*, and, as this name implies, it was introduced by Einstein for the discussion of extremely large-scale phenomena in astronomy. It is generally regarded as being so small (relative to other significant quantities of the same dimensions) that its effects are completely negligible in ordinary celestial mechanics, and even in cosmology its physical significance is doubtful. On the whole, as far as most of our work is concerned, the increased generality due to the inclusion of Λ seems outweighed by the increased complexity of the formulae, and so we shall drop it ¹ with the understanding that it may be reinstated when required in cosmological studies. Accordingly, until further notice the field equations read

$$G_{ij} = -\kappa T_{ij}, \quad \kappa = 8\pi. \quad (111)$$

In vacuo we delete the energy tensor, so that we have

$$G_{ij} = 0; \quad (112)$$

these equations are easily seen to be equivalent to

$$R_{ij} = 0. \quad (113)$$

These relativistic field equations bear no obvious resemblance to any equations of Newtonian theory. But we shall now establish a close connection by considering, in Newtonian theory and in relativity, the motion of a fluid without pressure, i.e. the incoherent fluid or dust cloud of § 4. In both cases times and lengths will be measured in seconds [$1 \text{ sec} = 2.998 \times 10^{10} \text{ cm}$, as in III-(134)]; Newtonian mass will be measured in grams and relativistic mass in seconds.

We start with the Newtonian equations, writing ρ for density, u_α for

¹ PAULI [1958, p. 220] has recommended that it should be dropped altogether, following EINSTEIN'S [1931] later view. Cf. EINSTEIN [1955, p. 127].

velocity, ϕ for gravitational potential, and γ for the gravitational constant. We have then the dynamical equations

$$\rho \left(\frac{\partial u_\alpha}{\partial t} + u_{\alpha,\gamma} u_\gamma \right) = \rho \phi_{,\alpha}, \quad (114)$$

the equation of mass-conservation

$$(\rho u_\alpha)_{,\alpha} = 0, \quad (115)$$

and Poisson's equation

$$\phi_{,\alpha\alpha} = -4\pi\gamma\rho. \quad (116)$$

In order to make comparison with relativity, we must deal with quantities which are physically observable without comparison with a hypothetical non-accelerated frame. In this sense $u_{\alpha,\beta}$ is physically observable, whereas $\partial u_\alpha / \partial t$ is not. Accordingly let us calculate the rate of change of $u_{\alpha,\beta}$ following the fluid: the value is

$$\frac{d}{dt} u_{\alpha,\beta} = \frac{\partial}{\partial t} u_{\alpha,\beta} + u_{\alpha,\beta\gamma} u_\gamma. \quad (117)$$

If we divide (114) by ρ and take the partial derivative with respect to x_β , we get

$$\frac{\partial}{\partial t} u_{\alpha,\beta} + u_{\alpha,\gamma\beta} u_\gamma + u_{\alpha,\gamma} u_{\gamma,\beta} = \phi_{,\alpha\beta}, \quad (118)$$

and so

$$\frac{d}{dt} u_{\alpha,\beta} = -u_{\alpha,\gamma} u_{\gamma,\beta} + \phi_{,\alpha\beta}. \quad (119)$$

We turn now to Einstein's field equations (111), which read, for an incoherent fluid,

$$G_{ij} = -\kappa\mu V_i V_j, \quad V_i V^i = -1, \quad (120)$$

where μ is the density and V^i the 4-velocity. The analogue of $u_{\alpha,\beta}$ is $V_{i|j}$, and we shall calculate its rate of change along a stream-line. We have already seen in § 4 that the stream-lines for an incoherent fluid are geodesics, so that we have

$$V_{i|k} V^k = 0, \quad V_{i|kj} V^k + V_{i|k} V_{|j}^k = 0. \quad (121)$$

Thus, following a stream-line and using I-(94), we have

$$\frac{\delta}{\delta s} (V_{i|j}) = V_{i|jk} V^k = V_{i|kj} V^k + R_{mijk} V^m V^k \quad (122)$$

or

$$\frac{\delta}{\delta s}(V_{i|j}) = -V_{i|k}V_{|j}^k + R_{mijk}V^mV^k. \quad (123)$$

Although (119) is set in three dimensions and (123) in four, we detect a strong formal resemblance, the term in (119) which involves the Newtonian potential being replaced by the curvature term in (123). We can bring the two equations closer together by using the rate-of-strain and the spin of the fluid, for the definitions of these quantities in § 3 are essentially the same as their definitions in Newtonian hydrodynamics. Thus for the Newtonian rate-of-strain and spin (or vorticity) we write

$$\sigma'_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}), \quad \omega'_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} - u_{\beta,\alpha}), \quad (124)$$

and (119) gives

$$\begin{aligned} \frac{d}{dt} \sigma'_{\alpha\beta} &= -\sigma'_{\alpha\gamma}\sigma'_{\gamma\beta} - \omega'_{\alpha\gamma}\omega'_{\gamma\beta} + \phi_{,\alpha\beta}, \\ \frac{d}{dt} \omega'_{\alpha\beta} &= -\sigma'_{\alpha\gamma}\omega'_{\gamma\beta} + \sigma'_{\beta\gamma}\omega'_{\gamma\alpha}. \end{aligned} \quad (125)$$

Since the absolute acceleration vanishes in the relativistic fluid, (62) and (63) give (we restore the sign for covariant differentiation)

$$\sigma_{ij} = \frac{1}{2}(V_{i|j} + V_{j|i}), \quad \omega_{ij} = \frac{1}{2}(V_{i|j} - V_{j|i}), \quad (126)$$

and (123) gives

$$\begin{aligned} \frac{\delta}{\delta s} \sigma_{ij} &= -\sigma_{ik}\sigma_{|j}^k - \omega_{ik}\omega_{|j}^k + R_{mijk}V^mV^k, \\ \frac{\delta}{\delta s} \omega_{ij} &= -\sigma_{ik}\omega_{|j}^k + \sigma_{jk}\omega_{|i}^k. \end{aligned} \quad (127)$$

Introducing a reference tetrad $\lambda_{(a)}^i$ with parallel transport along the stream-line and with $\lambda_{(4)}^i = V^i$, we write these equations in the form

$$\begin{aligned} \frac{d}{ds} \sigma_{(\alpha\beta)} &= -\sigma_{(\alpha\gamma)}\sigma_{(\gamma\beta)} - \omega_{(\alpha\gamma)}\omega_{(\gamma\beta)} + R_{(4\alpha\beta 4)}, \\ \frac{d}{ds} \omega_{(\alpha\beta)} &= -\sigma_{(\alpha\gamma)}\omega_{(\gamma\beta)} + \sigma_{(\beta\gamma)}\omega_{(\gamma\alpha)}. \end{aligned} \quad (128)$$

Since the distinction between t and s is rather trivial in the present connection, we must regard the agreement between the Newtonian equations (125) and the relativistic equations (128) as very close indeed; to make them agree, all we need is to connect the Newtonian potential and the Riemann tensor by the equations

$$\phi_{,\alpha\beta} = R_{(4\alpha\beta 4)}. \quad (129)$$

Thus if we have confidence in the practical validity of Newtonian theory as a guide to the interpretation of relativity, we are in a position to evaluate certain components of the Riemann tensor in terms of the derivatives of the Newtonian potential. Indeed we already took this step in III-(182).

But there is something more to be said. Inspection of the above calculations will show that we have used neither Poisson's equation (116) nor the Einstein equations (120) — all we need to establish (121) is the conservation equation $T^i{}_{|j} = 0$, which is a consequence of (120) but not by any means equivalent to it. Applying (116) to (129), by putting $\beta = \alpha$, we get

$$-4\pi\gamma\rho = R_{(44)} = R_{ij}V^iV^j. \quad (130)$$

Now (120) gives

$$G = g^{ij}G_{ij} = \kappa\mu, \quad (131)$$

and we have

$$\begin{aligned} R_{ij} &= G_{ij} - \tfrac{1}{2}g_{ij}G = -\kappa\mu V_iV_j - \tfrac{1}{2}g_{ij}\kappa\mu, \\ R_{ij}V^iV^j &= -\kappa\mu + \tfrac{1}{2}\kappa\mu = -\tfrac{1}{2}\kappa\mu = -4\pi\mu. \end{aligned} \quad (132)$$

Thus (130) establishes the following connection between the densities in the two theories:

$$\mu = \gamma\rho. \quad (133)$$

We are now in a position to find the number of seconds in a gramme. We know the value of the gravitational constant:

$$\gamma = 6.670 \times 10^{-8} \text{ g}^{-1}\text{cm}^3 \text{ sec}^{-2}. \quad (134)$$

Since, as in III-(134), $1 \text{ cm} = 3.336 \times 10^{-11} \text{ sec}$, we have

$$\gamma = 2.476 \times 10^{-39} \text{ g}^{-1} \text{ sec}. \quad (135)$$

Remembering that all lengths are measured in sec, we have density $= \mu \text{ sec}^{-2} = \rho \text{ g sec}^{-3}$. Let $1 \text{ g} = x \text{ sec}$. Then $\mu = x\rho$, and by (133)

$x = \gamma$, the numerical value (135). Therefore

$$\begin{aligned} 1 \text{ g} &= 2.476 \times 10^{-39} \text{ sec}, \\ 1 \text{ sec} &= 4.039 \times 10^{38} \text{ g}. \end{aligned} \tag{136}$$

The following masses may be noted:

$$\begin{aligned} \text{mass of electron} &= 2.255 \times 10^{-66} \text{ sec}, \\ \text{mass of moon} &= 1.813 \times 10^{-13} \text{ sec}, \\ \text{mass of earth} &= 1.479 \times 10^{-11} \text{ sec}, \\ \text{mass of sun} &= 4.920 \times 10^{-6} \text{ sec} \\ \text{mass of average galaxy}^1 &= 10^{44} \text{ g} = 2 \times 10^5 \text{ sec}. \end{aligned} \tag{137}$$

With mass and distance expressed in sec, force is dimensionless. The Newtonian attraction between masses m , m' at distance r is mm'/r^2 ; we have in fact made the gravitational constant unity.

§ 6. SURVEY OF FIELD EQUATIONS AND COORDINATE CONDITIONS

Enough has been said in the preceding section to indicate that, different though they look, the relativistic field equations and the equations of Newtonian hydrodynamics-cum-attractions say very nearly the same thing, at least in the case of a fluid without pressure. Since the basic concepts of Newtonian theory are so unsatisfactory we would pollute relativity by making it rest in any way on Newtonian ideas, and the preceding comparison was undertaken only to give confidence that we may expect to extract physically valid results from Einstein's field equations. These equations are very complicated mathematically, and our next step should be to survey them in general terms without at present becoming involved in technical complications. We recall from (111)–(113) that the field equations read

$$G_{ij} = -\kappa T_{ij}, \quad \kappa = 8\pi, \tag{138}$$

reducing in vacuo to

$$G_{ij} = 0 \text{ or equivalently } R_{ij} = 0. \tag{139}$$

We may approach these equations in three different spirits, realistic,

¹ BONDI [1952, p. 38].

agonistic and creative. The *realist* wants to connect the field equations with his already extensive knowledge of the physical universe. The *agonist*¹ wants to wrestle with difficult mathematical problems arising out of the field equations. The *creator's* pleasure lies in the construction of universes, fantastic or realistic, satisfying the field equations. In practice, these three ambitions are merged in the common quest for understanding, but the analysis is useful.

Of the solar system the realist knows that there is a dominating fluid body (the sun) with a number of planets (solid or fluid or both) circulating about it. These planets have satellites, and all the bodies spin. Except for very minor bodies, the intermediate space is empty of matter and any radiation there may be is not very significant dynamically. Thus inside the sun and the planets we are to apply the equations (138) and in the intermediate region the equations (139), with provision for junction conditions [cf. I-§ 9] at the surfaces of the sun and the planets. But the realist may prefer to consider, for simplicity, a universe in which there are only two bodies. In the space-time picture, these bodies appear as two timelike world-tubes (Fig. 4). Inside the tubes we have (138) and outside them (139). But what is T_{ij} inside the tubes? Let us for clarity recall what has been said about the energy tensor. We assume that it possesses four eigenvalues θ and four corresponding unit eigenvectors λ^i satisfying

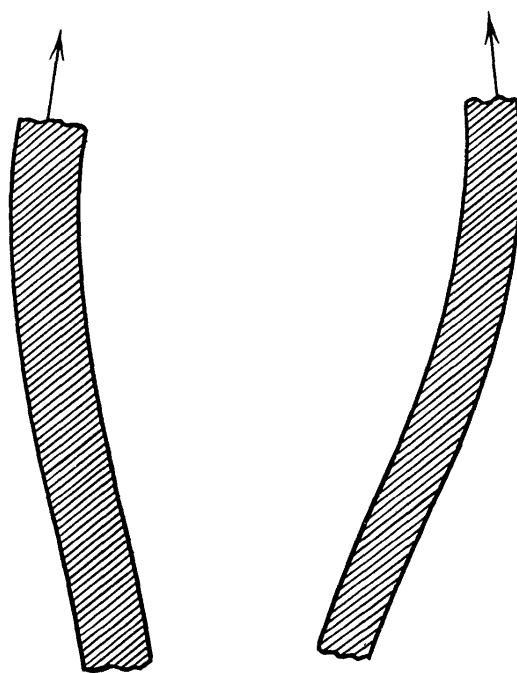


Fig. 4 – The two-body problem

$$T_{ij}\lambda^j = \theta\lambda_i. \quad (140)$$

We denote the timelike unit eigenvector by V^i and the corresponding eigenvalue by $-\mu$, so that we have

$$T_{ij}V^j = -\mu V_i. \quad (141)$$

¹ Used with the meaning of the Greek $\alpha\gamma\omega\nu\sigma\tau\acute{\eta}\varsigma$ = a combatant in the games, a contender for prizes.

If the other three eigenvalues are $\theta_{(\alpha)}$ and the corresponding unit (spacelike) eigenvectors are $\lambda_{(\alpha)}^i$, we can write T_{ij} in the form [cf. (78)]

$$T_{ij} = \mu V_i V_j - S_{ij}, \quad (142)$$

where S_{ij} is the stress tensor

$$S_{ij} = - \sum_{\alpha=1}^3 \theta_{(\alpha)} \lambda_{(\alpha)i} \lambda_{(\alpha)j}, \quad (143)$$

satisfying

$$S_{ij} V^j = 0. \quad (144)$$

We certainly expect the density μ to be positive. As regards stress, the preference for pressure over tension is less obvious, but on the astronomical scale we do not expect to find bodies under tension, because they would be pulled apart. Now $-\theta_{(\alpha)}$ are the three principal stresses; we want these negative, i.e. $\theta_{(\alpha)}$ positive, and so we may say that, for physical reasons, we want the eigenvalues of T_{ij} to have the signs

$$\text{eigenvalues of } T_{ij}: (+ + + -). \quad (145)$$

Note that, on account of the minus sign in (138), this implies ¹

$$\text{eigenvalues of } G_{ij}: (- - - +). \quad (146)$$

We see then that inside the world-tubes we have the field equations

$$G_{ij} = -\kappa \mu V_i V_j + \kappa S_{ij}, \quad (147)$$

and if the body is a perfect fluid, this reads

$$G_{ij} = -\kappa(\mu + p) V_i V_j - \kappa p g_{ij}; \quad (148)$$

we have also

$$V_i V^i = -1. \quad (149)$$

If n^i is the unit normal to the 3-space bounding a world-tube, the junction condition reads

$$T_{ij} n^j = 0, \quad (150)$$

¹ In view of the negative sign in the field equations (138) and the negative sign in the matrix $\eta_{ij} = \text{diag}(1, 1, 1, -1)$, it is easy to get confused in the matter of signs. Since the positive sign of the density μ is of considerable physical importance, we note that, if we take the coordinates in the usual way, with the parametric lines of x^α spacelike and that of x^4 timelike, then $\mu > 0$ requires

$$G_{44} < 0, \quad G_4^4 > 0, \quad T_{44} > 0, \quad T_4^4 < 0. \quad (146a)$$

These signs are easily checked from (147) and (138).

which means that n^i is an eigenvector with zero eigenvalue. Thus one of the principal stresses vanishes, n^i being the corresponding principal direction. Since V^i must be orthogonal to this direction, we have, as junction conditions,

$$V_i n^i = 0, \quad S_{ij} n^j = 0. \quad (151)$$

In the case of a perfect fluid, these read

$$V_i n^i = 0, \quad p = 0. \quad (152)$$

So much for the realist. The agonist starts in a primitive way by counting the differential equations and the unknowns which are to satisfy them. But here we must introduce the important idea of *coordinate conditions*. Within the condition of admissibility [I-§ 1], we have liberty in the choice of coordinates. We might use normal Gaussian coordinates, so that, as in I-(213),

$$g_{\alpha 4} = 0, \quad g_{44} = -1, \quad (153)$$

if we make the fourth coordinate timelike. These are *coordinate conditions* of a special type. They may be of a more general nature, but their essential feature is that they are *four* in number; the number *four* arises from the fact that a coordinate transformation involves *four* functions, expressing the new coordinates in terms of the old. Although normal Gaussian coordinates are probably the simplest special coordinates in space-time, there are also *null coordinates* for which¹

$$g^{11} = g^{22} = g^{33} = g^{44} = 0; \quad (154)$$

if we start from any coordinates \bar{x}^i and contravariant metric tensor \bar{g}^{ij} , we obtain coordinates x^a for which (154) hold by solving the four partial differential equations of the type

$$\bar{g}^{ij} \frac{\partial x^1}{\partial \bar{x}^i} \frac{\partial x^1}{\partial \bar{x}^j} = 0. \quad (155)$$

A number of coordinate conditions occur in the literature of relativity, designed for special purposes. To put the matter in general form, we shall denote the coordinate conditions by

$$C_i = 0; \quad (156)$$

these are equations (perhaps differential equations) satisfied by the

¹ Cf. SYNGE [1960a].

metric tensor g_{ij} . They are of course *not* tensor equations, since they are satisfied only when the coordinates are specially chosen.

We are now prepared to make a count. Taking the case of a perfect fluid, inside the world tubes we have, in (148), (149) and (156), $10 + 1 + 4 = 15$ equations to be satisfied by the following 16 unknowns: g_{ij} , V_i , μ , p . The system is indeterminate. To make it determinate, we need some additional hypothesis, such as is made in Newtonian hydrodynamics: we might assume μ constant, or we might set up some rather ad hoc equation connecting μ and p . As for the field in vacuo, we have in (139) apparently 10 equations; but on account of the identity

$$G^i_{|j} = 0 \quad (157)$$

there are actually only 6 independent equations in (139). With the 4 equations in (156), we see 10 equations for the 10 unknowns g_{ij} . Thus, as far as the mere counting of equations goes, we have a determinate problem (e.g. in the case of two bodies) if we add some pressure-density equation.

But it is far cry from *counting* partial differential equations to *solving* them. Faced with such a complicated situation, we pause to ask ourselves what we are seeking. The readiest answer is that we are trying to do in relativity what has been done in Newtonian dynamics. Now Newtonian dynamics has been remarkably successful in celestial mechanics, and we may ask whether that success has been due to some subsidiary assumptions validated by common sense. In the case of the two-body problem, an exact treatment of two *fluid* masses is too complicated, and it is usual to think only of two *rigid bodies*. Further, the bodies may be taken to be spherical, and indeed they may be reduced to two massive particles — then the two-body problem becomes an elementary exercise. But the essence of the Newtonian success lies in the reduction of *partial* differential equations to *ordinary* differential equations by the assumption of *rigidity*.

When we try to apply the Newtonian simplifications to the relativistic two-body problem we find that they will not work. The concept of rigidity, so fundamental in Newtonian theory, does not lend itself to relativity, and the reduction of the bodies to point-particles would take us outside relativity, which is essentially a field theory. These facts do not of course rule out judicious use of approximations based on reasonable assumptions that the rate-of-strain is very small or that the

size of each body is very small. There are also two other bases of approximation: (i) all known gravitational fields are weak (small curvature of space-time), and (ii) relative velocities are small.

Having dealt with the realist and the agonist, we shall now describe a creator's plan for building universes consistent with the field equations. Some of these may turn out to be very queer indeed, but, if we have confidence in the physical validity of Einstein's theory, we know that the whole class of universes created in this way contains one of particular interest to us — the universe which we actually inhabit.

Any set of ten functions $g_{ij}(x)$, sufficiently smooth (let us for simplicity suppose them of class C^2), define a Riemannian space-time if $g_{ij}dx^i dx^j$ has the correct signature, i.e. if g_{ij} is locally reducible to $\text{diag}(1, 1, 1, -1)$. If then we choose such functions arbitrarily, we have, by (138), a universe in which the energy tensor is

$$T_{ij} = -\kappa^{-1}G_{ij}, \quad (158)$$

the Einstein tensor having been calculated from g_{ij} — this involves no more than finding g^{ij} algebraically and carrying out the required differentiations. There are no partial differential equations to solve. Since the procedure is based on chosen values of g_{ij} , we shall call it the *g-method*. The crude Newtonian analogue would be to explore gravitation by assuming the potential ϕ and calculating the density ρ by means of Poisson's equation, written in the form

$$\rho = -(4\pi\gamma)^{-1}\Delta\phi. \quad (159)$$

The plan looks suspiciously simple, and it is subject to two criticisms. First, the universes obtained blindly in this way are not likely to bear any resemblance at all to the universe as we know it. Secondly, until we have carried out the calculations, we do not know the signs of the eigenvalues of T_{ij} , and unless these turn out to be $(+ + + -)$ as in (145), it is an unnatural universe. Of these signs the last is the most important. If the fourth eigenvalue of T_{ij} is positive (i.e. the last eigenvalue of G_{ij} negative), we have negative density, and while in modern physics negative density may not seem as outrageous as it did once, it is not realistic as far as celestial mechanics is concerned.

But however slight the practical value of such created universes may be, one learns much from baffled attempts to construct in this way models resembling, even remotely, the universe as we know it. Since,

in the matter of density, only one sign is involved, we might expect, with a random choice of g_{ij} (subject only to the signature condition), a probability $\frac{1}{2}$ of getting a positive density, and a probability $1/16$ of getting all four eigenvalues with the required signs. But it is not as simple as that, for we need the correct signs, not at a single event, but throughout space-time. Anyone who carries out experimental calculations finds that the scales are weighted against him — the density shows a perverse tendency to become negative in parts of space-time.

In view of the connection established in § 5 between Newtonian theory and relativity, light is thrown on this question of the fluctuating sign of density by studying Poisson's equation in the form (159). By virtue of the theorem of Gauss, the flux across any closed surface of the normal gradient of ϕ is proportional to the total mass inside the surface. If we make ϕ tend to zero at infinity more rapidly than $1/r$, the total mass of the system is zero, and so the sign of ρ must fluctuate between positive and negative; if ϕ does not tend to zero at least as rapidly as $1/r$, we are likely to get an infinite total mass. It is clear that the behaviour of ϕ at infinity is critical, and that (to put the matter in a rough but dramatic form) the probability that density will fluctuate in sign is much greater than $\frac{1}{2}$.

As we shall see later, there is a relativistic analogue of the theorem of Gauss (unfortunately much more complicated) and it likewise indicates that a fluctuating sign in density is to be expected from an arbitrary choice of g_{ij} . For that reason, however amusing it may be to construct in this way fantastic universes, some rather severe guidance is needed if anything like the natural universe is to emerge, and it is by no means easy to prescribe the rules.

Having now viewed the field equations as realist, as agonist, and as creator, the reader may conclude that, when compared with Newtonian theory, relativity offers us a view of the universe which is obscure and confused. He would not be far wrong. We have not really mastered Einstein's theory yet, and this should make us appreciate all the more those exact solutions of the field equations which have been found.

But let us push the general study of the field equations a little further. In the g -method, g_{ij} are given and (138) simply defines T_{ij} . Reversing the roles, we now regard T_{ij} as given (T -method), so that

$$G_{ij} = -\kappa T_{ij} \quad (160)$$

is a set of ten non-linear second-order partial differential equations to be satisfied by g_{ij} . No coordinate conditions may be added. The four conservation equations

$$g^{jk}T_{ij|k} = 0 \quad (161)$$

are consequences of (160) and imply no restriction on the chosen T_{ij} , since they contain the unknown g_{ij} , not only in the coefficients but also in the covariant derivatives.

Just as the g -method has a Newtonian analogue as indicated in (159), the analogue in the T -method is the problem of finding the potential ϕ of a given distribution of mass. The solution is an integral which, in the case of concentrated masses, yields the familiar expression

$$\phi = \sum \frac{\gamma^m}{r}. \quad (162)$$

To this solution may be added any harmonic function, but it is usually ruled out by the demand that the potential should vanish at infinity.

To the realist, the T -method is more pleasing than the g -method, because negative density may be avoided by choosing T_{ij} properly. But the method fails in realism through disregard of the physical constitution of matter, as we see on reference to the realistic equation (148) for a perfect fluid. To the agonist, the T -method offers a problem of stimulating difficulty in the domain of non-linear partial differential equations, and three major lines of attack have been developed:

- (i) Conditions of symmetry are imposed, reducing drastically the numbers of independent variables and unknowns (see Chaps. VII–IX).
- (ii) Attention is directed to the Cauchy (initial-value) problem (see Chap. v).
- (iii) A method of successive approximations is used.

To start a method of successive approximations, we note that, if T_{ij} were zero, we would have a solution

$$g_{ij} = \eta_{ij} = \text{diag}(1, 1, 1, -1), \quad (163)$$

the metric tensor of flat space-time. Regarding T_{ij} as small, we then write

$$g_{ij} = \eta_{ij} + g_{ij}^{(1)} + g_{ij}^{(2)} + \dots \quad (164)$$

the numerical labels referring to order of magnitude in terms of T_{ij} . Substituting in (160) and separating the terms of different orders, we

get partial differential equations for the several terms in (164). The $g_{ij}^{(1)}$ is called the *linear approximation* because the corresponding partial differential equations are linear. Subject to a certain restriction on the choice of T_{ij} , the linear approximation will be worked out in the next chapter; it corresponds to the Newtonian formula (162). On going beyond the linear approximation, complications accumulate to such an extent that it is very hard to assess the mathematical or physical meaning of the results so far obtained by the exercise of much ingenuity and perseverance ¹. We shall now describe a less ambitious two-step plan, suggested by the linear approximation just mentioned and the following two thoughts:

- (i) In a complicated situation such as we have before us, an exact solution of the field equations is to be esteemed far above any approximation, even though the exact mathematical solution is admittedly only an approximation to physical reality (no mathematical formula is ever more than this anyway).
- (ii) We should not regard (160) as a set of equations to be solved for g_{ij} but as a set of equations to be satisfied by the 20 quantities g_{ij} , T_{ij} ; any solution is precious, but particularly those which come near to physical reality.

The plan is a mixture of the T -method and the g -method, and it involves the following steps:

a) Choose any set of symmetric functions $A_{ij}(x)$, satisfying the four conditions

$$\eta^{jk}A_{ij,k} = 0. \quad (165)$$

(Note that these are partial derivatives!)

b) Substitute

$$g_{ij} = \eta_{ij} + \gamma_{ij} \quad (166)$$

in I-(108) and carry out a formal calculation of G_{ij} as if γ_{ij} were infinitesimal, retaining only the principal part; the result, denoted by

¹ EINSTEIN, INFELD and HOFFMANN [1938], EINSTEIN and INFELD [1940], [1949], INFELD [1953], [1954a, b], [1955d], [1957a, b], [1959], INFELD and PLEBANSKI [1956a, b], INFELD and SCHEIDEGGER [1951], BERGMANN [1942], BONNOR [1959a], B. FINZI [1959]. These references deal with the so-called 'problem of motion'; the method is not quite the T -method as described above, because only the vacuum equations (139) are used and matter appears as singularities. See also FOCK [1939a, b], [1941], [1950], [1955b], [1959], and papers by CLARK listed in bibliography.

H_{ij} , is

$$H_{ij} = \frac{1}{2}\{\eta^{ab}(\gamma_{ab,ij} + \gamma_{ij,ab} - \gamma_{ai,bj} - \gamma_{aj,bi}) - \eta_{ij}\eta^{ab}\eta^{cd}(\gamma_{ab,cd} - \gamma_{ac,bd})\}. \quad (167)$$

c) Write down the equations

$$H_{ij} = -\kappa A_{ij}, \quad (168)$$

and regard these as partial differential equations for γ_{ij} (T -method). These equations are consistent by virtue of (165), and they possess a particular solution of great interest and simplicity, commonly called that of the retarded (or advanced) potential. This will be discussed in detail in the next chapter. Here it suffices to say that this particular solution γ_{ij} vanishes if we put $A_{ij} = 0$.

d) The particular solution is now put into (166), so that we get g_{ij} , and the corresponding G_{ij} is calculated. Then the energy tensor is obtained by (g -method)

$$T_{ij} = -\kappa^{-1}G_{ij}, \quad (169)$$

and we have a set g_{ij} , T_{ij} satisfying the field equations (160).

In order that the above procedure may work it is necessary to impose on A_{ij} , in addition to the equations (165), only conditions of smoothness and of vanishing sufficiently rapidly at infinity, and we can of course choose A_{ij} suitably. But we have to examine two things: (i) the signature of g_{ij} , and (ii) the signs of the eigenvalues of T_{ij} . These are both taken care of by making A_{ij} small enough. There is no question here of making A_{ij} infinitesimal, for the conditions of signature and signs are in the nature of inequalities. By making A_{ij} small enough, we can make γ_{ij} small enough to give g_{ij} , as in (166), the proper signature. The question of the eigenvalues is a little more delicate. Let us suppose that A_{ij} is chosen so that its eigenvalues relative to η_{ij} have the natural signs, i.e. so that the roots of the determinantal equation

$$\det |A_{ij} - \theta \eta_{ij}| = 0 \quad (170)$$

are $(+ + + -)$ [cf. (145)]. The eigenvalues of T_{ij} are the values of θ' satisfying

$$\det |T_{ij} - \theta' g_{ij}| = 0. \quad (171)$$

Now A_{ij} are chosen small, say O_1 . Then $\gamma_{ij} = O_1$, so that g_{ij} differs

from η_{ij} by O_1 . When we evaluate G_{ij} , we get

$$G_{ij} = H_{ij} + O_2, \quad (172)$$

and so, by (169),

$$T_{ij} = A_{ij} + O_2. \quad (173)$$

It follows then that, in passing from (170) to (171), we do not (provided A_{ij} is small enough) disturb the signs of the eigenvalues; nor will we disturb much the directions of the eigenvectors.

But it must be emphasized that we are *not* making any approximation. No small quantities are rejected. All we need is to make A_{ij} *small enough, but still definitely finite*. Such a method may be called a *feed-back method*, because pilot-values obtained from a linear approximation are fed back into the exact equations. The method will be used in the next chapter.

§ 7. NOTE ON THE MOTION OF AN ISOLATED BODY

Let us study realistically the motion of a material body which is isolated in the sense that it is moving in vacuo. Other material bodies may be present, but we are not concerned with them. Inside the world-tube of the body we have the field equations

$$G_{ij} = -\kappa T_{ij}, \quad \kappa = 8\pi, \quad (174)$$

and outside that tube we have

$$G_{ij} = 0. \quad (175)$$

Let Σ be the boundary of the world-tube. Then G_{ij} is discontinuous across Σ , the discontinuity being however subject to the junction condition 1-(229), which imposes on G_{ij} inside the tube the condition

$$G_{ij}N^j = 0 \text{ on } \Sigma, \quad (176)$$

where N^i is the unit outward normal to Σ . This condition implies that on Σ the Einstein tensor has one zero eigenvalue, N^i being the corresponding eigenvector. Since eigenvectors are mutually orthogonal, the timelike eigenvector of G_{ij} (i.e. the 4-velocity V^i) is orthogonal to N^i , so that, as in (151),

$$V_i N^i = 0 \text{ on } \Sigma. \quad (177)$$

This equation says in fact that Σ is composed of stream-lines, a statement which we might well be prepared to accept on its own merits.

However, it should be realized that the isolation imposed by (176) is very strict; physically, it implies that there is no radiation from the body, i.e. no flux of energy across its surface.

In view of (174), it is obviously a matter of indifference whether we work with G_{ij} (geometry) or T_{ij} (physics). Let us work with T_{ij} , recalling that

$$T_{ij} = \mu V_i V_j - S_{ij}, \quad (178)$$

where μ is density, V_i 4-velocity and S_{ij} stress, satisfying

$$S_{ij} V^j = 0. \quad (179)$$

The basic equation for present purposes is (105):

$$\mu D V^i = S^i_j{}_{|j} - V^i S^{jk} \sigma_{jk}; \quad (180)$$

here $D = \delta/\delta s$ (the absolute derivative along a stream-line) and σ_{jk} the rate-of-strain tensor (62), satisfying

$$\sigma_{jk} V^k = 0. \quad (181)$$

We note that (180) expresses the absolute acceleration of a stream-line (equivalently, its first curvature vector) in terms of other quantities, and the condition that a stream-line should be geodesic is

$$D V^i = 0. \quad (182)$$

Now the argument which is being given here is motivated by the desire to throw light on the geodesic hypothesis (III-§ 3), and we are tempted to try to force (182) out of (180) by some limiting process. We can in fact do so very easily by assuming that there is no stress in the body. But this would amount merely to what has been done already in § 4, where it was shown that in an incoherent fluid (or dust cloud) the stream-lines are geodesic. The case of vanishing stress is not interesting physically, however, and an approximate argument for the case of small stress is likely to become confused, since stress is a dimensional quantity and can be called small only in comparison with other quantities of the same dimensions. We shall therefore abandon the idea of approximation and see what can be deduced accurately from (180).

It is convenient to define, at each event on the boundary Σ of the world-tube, a quantity

$$Q = N_i S^i_j{}_{|j}. \quad (183)$$

Then, by virtue of (177), (180) gives on Σ

$$\mu N_i DV^i = Q. \quad (184)$$

In general the motion of the body is rotational in the sense of § 3; in other words, the body has spin. However the irrotational case (no spin) is much easier to discuss and we shall treat it first.

In irrotational motion the stream-lines form a normal congruence, and we can draw a normal 3-dimensional section S of the world-tube (Fig. 5). Let σ be the 2-dimensional intersection of S and Σ . Then N^i , the outward unit normal to Σ , is at the same time the outward unit

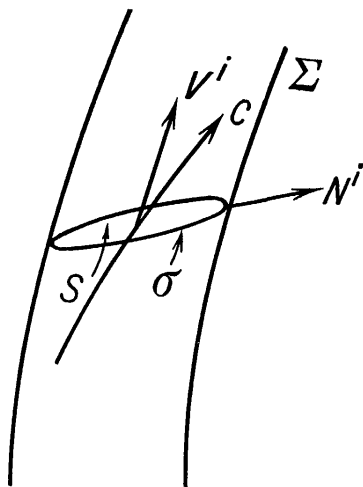


Fig. 5 – World-tube
without spin

normal to σ in the 3-space S . Since DV^i is orthogonal to V^i , DV^i lies in S , and so the invariant $N_i DV^i$ is the outward component of DV^i taken on the normal to σ in S . Taking, as is natural, the density to be positive, we see from (184) that the sign of this outward normal component is the sign of Q . In fact, if

$$Q > 0 \text{ over } \Sigma, \quad (185)$$

then DV^i points out of σ everywhere on σ . But the fixed-point theorem tells us that if we have a spacelike 3-space with a vector field pointing *out* all over the boundary of

that 3-space, then the vector vanishes somewhere in the 3-space. We conclude then that, *if (185) holds, there exists some point of S at which (182) holds*. Since the world-tube has a single infinity of normal sections, we have then inside the tube a single infinity of points at each of which (182) holds. These points form a curve C inside the tube (Fig. 5). Note that C is a *locus of no-acceleration*. It is not in general itself a stream-line — indeed there is no reason to suppose C timelike. We shall return to the condition (185) below, interpreting it in the case of a fluid.

We pass to the more complicated case of a *spinning body*. Now the world-tube does not possess normal sections, and we are at a loss what sections to take in order to apply the type of argument used above. If we take any spacelike section S , with unit future-pointing normal n^i , then on S we have

$$-K \leq n_i V^i \leq -1, \quad (186)$$

where K is the maximum value of $|n_i V^i|$ on S . In the case of irrotational motion, we can make $K = 1$ (by making S a normal section), and in general we would like to choose S so as to make K as small as possible on Σ . We shall not attempt to discuss that interesting geometrical problem; we shall be satisfied to accept the existence of some K such that on Σ

$$1 \leq |n_i V^i| \leq K. \quad (187)$$

In a vague way, we may regard the excess of K over unity as a measure of the spin of the body.

Fig. 6 shows Σ , the boundary of the world tube, with unit outward normal N^i ; S , the spacelike section, with unit normal n^i ; σ , the intersection of S and Σ ; v^i , the unit outward normal to σ in S ; and V^i , the 4-velocity. The bound K in (187) puts a bound on $|n_i N^i|$, and this may be investigated by using, at any point on σ , coordinates such that

$$\begin{aligned} g_{ij} &= \text{diag}(1, 1, 1, -1), \\ V^4 &= 1, \quad N^1 = 1, \end{aligned} \quad (188)$$

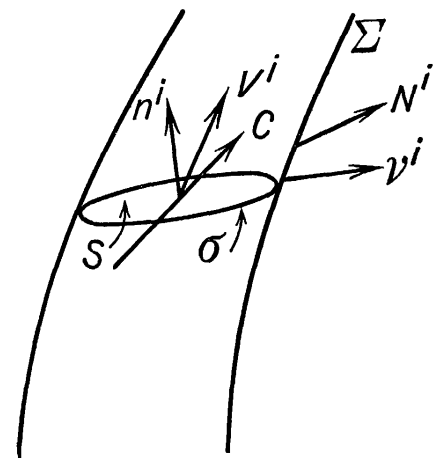


Fig. 6 - World-tube with spin

the other components of V^i and N^i vanishing. Then, by (187),

$$|n_4| \leq K. \quad (189)$$

Hence

$$n_1^2 = n_4^2 - 1 - n_2^2 - n_3^2 \leq K^2 - 1, \quad (190)$$

and so we have the required bound,

$$|n_i N^i| \leq (K^2 - 1)^{\frac{1}{2}}. \quad (191)$$

Similarly, using the fact that

$$V_i D V^i = 0, \quad (192)$$

we can show that

$$|n_i D V^i| \leq b(K^2 - 1)^{\frac{1}{2}}, \quad (193)$$

where b is the first curvature of a stream-line (the absolute acceleration), given by

$$b^2 = g_{ij} D V^i D V^j. \quad (194)$$

Since v^i points outwards and $v_i dx^i = 0$ for every displacement satisfying $n_i dx^i = 0$ and $N_i dx^i = 0$, we have

$$v^i = \alpha N^i + \beta n^i, \quad \alpha > 0. \quad (195)$$

Hence, since $n_i v^i = 0$ and $n_i n^i = -1$,

$$v^i = \alpha(N^i + n^i n_j N^j), \quad \alpha > 0. \quad (196)$$

Let W^i be the orthogonal projection of DV^i on S . The outward component of W^i on the normal v^i of σ is

$$W_i v^i = v_i DV^i = \alpha(N_i DV^i + n_i DV^i n_j N^j). \quad (197)$$

In order to be able to apply the fixed-point theorem, we now impose (in addition to (187)) an inequality which ensures that the expression (197) is positive all over σ . Let us assume that

$$Q > \mu b(K^2 - 1) \quad (198)$$

all over σ . Then, by (184), (191) and (193), $W_i v^i$ is positive all over σ , and so W^i must vanish somewhere in S . Now $W^i = 0$ implies that DV^i is either orthogonal to S or vanishes. But DV^i , being spacelike, cannot be orthogonal to S . Therefore, *if (187) and (198) hold all over σ , there exists a point in S at which $DV^i = 0$* . Hence we get, as in the irrotational case, a locus C of no-acceleration.

Let us examine the condition (198) which ensures the existence of a locus of no-acceleration. By (180) we have

$$b^2 \mu^2 = S_{|j}^{ij} S_{i \cdot |k}^{\cdot k} - 2V_i S_{|j}^{ij} S^{mn} \sigma_{mn} - (S^{mn} \sigma_{mn})^2. \quad (199)$$

But

$$V_i S_{|j}^{ij} = (V_i S^{ij})_{|j} - V_{i|j} S^{ij} = -\sigma_{ij} S^{ij}, \quad (200)$$

and so

$$b^2 \mu^2 = S_{|j}^{ij} S_{i \cdot |k}^{\cdot k} + (S^{ij} \sigma_{ij})^2. \quad (201)$$

Hence the condition (198) is equivalent to

$$N_i S_{|j}^{ij} > (K^2 - 1)[S_{|j}^{ij} S_{i \cdot |k}^{\cdot k} + (S^{ij} \sigma_{ij})^2]^{\frac{1}{2}}. \quad (202)$$

This condition as it stands is rather too complicated to be interesting. But in the case of a perfect fluid we have

$$S_{ij} = -p(V_i V_j + g_{ij}), \quad (203)$$

and the pressure p vanishes on Σ . Hence on Σ we have

$$\begin{aligned} S_{ij} &= 0, \\ S_{|j}^{ij} &= -p_{,j}(V^i V^j + g^{ij}) = -g^{ij} p_{,j}, \end{aligned} \quad (204)$$

and (202) simplifies to

$$-p_{,i}N^i > (K^2 - 1)(g^{ij}p_{,i}p_{,j})^{\frac{1}{2}}. \quad (205)$$

But it is really simpler still. For, since $p = 0$ on Σ , we have

$$(g^{ij}p_{,i}p_{,j})^{\frac{1}{2}} = |p_{,i}N^i|, \quad (206)$$

and so (205) is equivalent to the pair of conditions

$$p_{,i}N^i < 0, \quad K^2 < 2. \quad (207)$$

If we agree that pressure is necessarily positive, p must increase as we pass in through Σ , and so the first of (207) is necessarily satisfied in any physical situation. The second inequality in (207) is satisfied if the body is not spinning too rapidly (to put it rather vaguely). *Under these conditions the world-tube of a fluid body contains a curve of no-acceleration.*

A mathematical method which involves extraneous things is imperfect. The concept of no-acceleration has nothing to do with taking a section of the world-tube, and in fact this section is extraneous to the problem. It may be that there is a better and more direct way of investigating events of no-acceleration.

To come down to earth, literally, we may ask whether the world-tube of the earth contains a curve of no-acceleration. Probably it does. For the stress in the earth is largely hydrostatic pressure and its spin is small (or is it?). In any case if the geophysicist is satisfied that (207) are applicable to the earth and true for it, then certainly there is a curve of no-acceleration. But it must be understood that there is nothing in the argument indicating that the line of no-acceleration is the world-line of a particle of the earth.

We shall return to the isolated body in VI-§ 6.

CHAPTER V

SOME PROPERTIES OF EINSTEIN FIELDS

§ 1. THE BASIC FORMULA FOR RETARDED OR ADVANCED POTENTIAL

There is a theorem about retarded or advanced potential which is so important that it deserves a simple direct proof.

We recall, for a general space-time with metric tensor g_{ij} , the well known definition of the d'Alembertian operator applied to an invariant F :

$$\square F = g^{ij} F_{|ij} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^i} \left(\sqrt{-g} g^{ij} \frac{\partial F}{\partial x^j} \right). \quad (1)$$

In flat space-time with $g_{ij} = \eta_{ij} = \text{diag}(1, 1, 1, -1)$, we have

$$\square F = \eta_{ij} \frac{\partial^2 F}{\partial x^i \partial x^j}, \quad (2)$$

and this formula may be applied when F is a Cartesian component of a tensor; it yields a tensor of the same type. If we use curvilinear coordinates in flat space-time, the operator \square , applied to a Cartesian component, is as in (1); we transform the operator without changing F .

For polar coordinates in flat space-time the metric form is

$$dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 - dt^2, \quad (3)$$

and if we pass to coordinates $(u, v, \theta, \phi) = (x^1, x^2, x^3, x^4)$ by putting

$$r = \frac{1}{\sqrt{2}}(u - v), \quad t = \frac{1}{\sqrt{2}}(u + v), \quad (4)$$

so that

$$u = \frac{1}{2\sqrt{2}}(t + r), \quad v = \frac{1}{2\sqrt{2}}(t - r), \quad (5)$$

we get

$$\begin{aligned} g_{12} &= -1, & g_{33} &= r^2, & g_{44} &= r^2 \sin^2 \theta, \\ g^{12} &= -1, & g^{33} &= r^{-2}, & g^{44} &= r^{-2} \operatorname{cosec}^2 \theta, \\ \sqrt{-g} &= r^2 \sin \theta, & r^2 &= \frac{1}{2}(u^2 - 2uv + v^2), \end{aligned} \quad (6)$$

the other components of g_{ij} and g^{ij} vanishing. Then (1) gives

$$\square F = -2 \frac{\partial^2 F}{\partial u \partial v} + \frac{2}{u-v} \left(\frac{\partial F}{\partial u} - \frac{\partial F}{\partial v} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial F}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 F}{\partial \phi^2}. \quad (7)$$

We shall now calculate the integral

$$\int \square F d\omega \quad (8)$$

taken over a sheet of the null cone which has its vertex at $r = 0, t = 0$, $d\omega$ being the invariant element of 2-content¹. The past and future sheets of the null cone have the equations $u = 0$ and $v = 0$, respectively (Fig. 1), and it is clear from the symmetry of (7) in u and v that the formal calculations are the same for both. Let us integrate over $v = 0$, so that we have $r = u/\sqrt{2}$ and the element of 2-content is

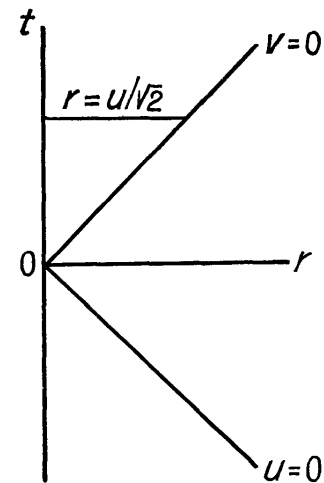


Fig. 1 – The two sheets of the null cone

$$d\omega = \frac{1}{r} \cdot r^2 \sin \theta dr d\theta d\phi = \frac{1}{2} u \sin \theta du d\theta d\phi. \quad (9)$$

It is clear from (7) that the integral is improper, so we cut out a small piece of the null cone at the vertex O and integrate as in (8) for the ranges

$$\varepsilon \leq u < \infty, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi < 2\pi. \quad (10)$$

Now the integrations with respect to θ and ϕ , applied to the last two terms in (7), give zero, and so, for the limits (10), we have

$$\int_{\varepsilon} \square F d\omega = \iint \sin \theta d\theta d\phi \int_{\varepsilon}^{\infty} \left\{ -\frac{\partial^2 F}{\partial u \partial v} + \frac{1}{u} \left(\frac{\partial F}{\partial u} - \frac{\partial F}{\partial v} \right) \right\} u du. \quad (11)$$

The last integral is

$$\begin{aligned} \int_{\varepsilon}^{\infty} \left(-u \frac{\partial^2 F}{\partial u \partial v} + \frac{\partial F}{\partial u} - \frac{\partial F}{\partial v} \right) du &= \int_{\varepsilon}^{\infty} \frac{\partial}{\partial u} \left(F - u \frac{\partial F}{\partial v} \right) du \\ &= \left[F - u \frac{\partial F}{\partial v} \right]_{u=\varepsilon}^{u=\infty}, \end{aligned} \quad (12)$$

¹ Cf. SYNGE [1956a, p. 430].

provided these limits exist. Thus, if F tends to zero at infinity in such a way that, along the null cone,

$$F - u \frac{\partial F}{\partial v} \rightarrow 0 \text{ as } u \rightarrow \infty, \quad (13)$$

we have

$$\int \square F d\omega = -4\pi F_0, \quad (14)$$

where F_0 is the value of F at the vertex of the null cone. This is the required result. It holds for integration over either sheet of the null cone.

If, in flat space-time, we seek a solution of the partial differential equation

$$\square H = F, \quad (15)$$

we may examine, as a tentative solution,

$$H(P') = -\frac{1}{4\pi} \int F(P) d\omega \quad (16)$$

where P is a current point on either sheet of the null cone having P' for vertex. To differentiate with respect to P' , we move the null cone and find the increments in the contributions from corresponding equal elements $d\omega$. This is equivalent to differentiating under the sign of integration with respect to P . Hence, using (14), we obtain

$$\square' H(P') = -\frac{1}{4\pi} \int \square F(P) d\omega = F(P'), \quad (17)$$

which shows that (16) is in fact a solution of (15). But we are not to think that it is a unique solution, because we can add to H any wave function, unless this addition should be ruled out by a subsidiary condition ¹.

§ 2. THE LINEAR APPROXIMATION

We shall now take the first step in generating Einstein fields according to the feed-back plan of IV-§ 6. The essential point of the argument is lost if we blunt the edge of the mathematics and speak vaguely of approximations, and yet the motivation springs from a rather natural

¹ Cf. SYNGE [1956a, pp. 361, 367] for a simple wave function which vanishes like $1/r^2$ at infinity and has no singularity. Cf. BONNOR [1957a].

approximation. Let us compromise by setting out the problem rather intuitively first, and then make a fresh start on a better mathematical basis.

Imagine some distribution of matter with small energy tensor T_{ij} producing a weak field. Then there exist coordinates x^i such that

$$g_{ij} = \eta_{ij} + \gamma_{ij}, \quad (18)$$

where γ_{ij} and its derivatives are small. We seek to solve the field equations

$$G_{ij} = -\kappa T_{ij}, \quad (19)$$

but we shall be satisfied by a linear approximation in which terms of the second degree in γ_{ij} are neglected. Thus we throw away these higher-order terms and reduce G_{ij} to H_{ij} as in IV-(167). It is then a question of solving a set of linear partial differential equations. We shall state the result below, but it is best at this point to turn to precise mathematics, since we need a precise result for later use and from it the intuitive answer can be extracted without difficulty.

Let there be a 4-space with coordinates x^i , ranging from $-\infty$ to $+\infty$. To be able to speak geometrically, we impose the Minkowskian metric tensor $\eta_{ij} = \text{diag}(1, 1, 1, -1)$, so that our space can be regarded as flat space-time and we can speak of null cones in it. We introduce symmetric functions $A_{ij}(x)$ satisfying

$$\eta^{jk} A_{ij,k} = 0, \quad (20)$$

and also satisfying conditions of smoothness and of vanishing at infinity (i.e. for $r \rightarrow \infty$ where $r^2 = x^\alpha x^\alpha$), these conditions (which we shall not trouble to specify in detail) being such that the operations described below can be carried out as required. We seek γ_{ij} to satisfy the equations

$$H_{ij} = -\kappa A_{ij}, \quad (21)$$

where H_{ij} is as in IV-(167). It is convenient to define the linear operator L_{ij}^{ab} by

$$\begin{aligned} L_{ij}^{ab} X_{ab} = & \eta^{ab} (X_{ab,ij} + X_{ij,ab} - X_{ai,bj} - X_{aj,bi}) \\ & - \eta_{ij} \eta^{ab} \eta^{cd} (X_{ab,cd} - X_{ac,bd}). \end{aligned} \quad (22)$$

Then (21) may be written

$$L_{ij}^{ab} \gamma_{ab} = -2\kappa A_{ij}; \quad (23)$$

these are the equations we seek to solve. Now comes the essential and curious part of the argument; if we write for brevity

$$\eta^{ab}A_{ab} = A, \quad (24)$$

then, by (20), we have (as is easy to verify by direct calculation)

$$L_{ij}^{::ab}(A_{ab} - \tfrac{1}{2}\eta_{ab}A) = \eta^{ab}A_{ij,ab} = \square A_{ij}, \quad (25)$$

where \square is the d'Alembertian operator. This is the key to the situation; as a tentative solution of (23) we try

$$\gamma_{ij}(P') = C \int (A_{ij} - \tfrac{1}{2}\eta_{ij}A) d\omega, \quad (26)$$

where C is a constant and the integration is taken over either sheet of the null cone having P' for vertex, $d\omega$ being the element of 2-content (remember that we are working in a flat space-time). We now apply the operator L , and differentiate under the integral sign, because that is equivalent to shifting the null cone. By (25) and (14) we get

$$\begin{aligned} L_{ij}^{::ab}\gamma_{ab}(P') &= C \int L_{ij}^{::ab}(A_{ab} - \tfrac{1}{2}\eta_{ab}A) d\omega \\ &= C \int \square A_{ij} d\omega = -4\pi C A_{ij}(P'). \end{aligned} \quad (27)$$

Thus (26) satisfies (23) provided

$$4\pi C = 2\kappa, \quad C = \kappa/2\pi. \quad (28)$$

We have then the following particular solution of (21):

$$\gamma_{ij}(P') = \frac{\kappa}{2\pi} \int (A_{ij} - \tfrac{1}{2}\eta_{ij}A) d\omega. \quad (29)$$

This holds for any κ ; if we put $\kappa = 8\pi$ as in IV-(110), we have

$$\gamma_{ij}(P') = 4 \int (A_{ij} - \tfrac{1}{2}\eta_{ij}A) d\omega. \quad (30)$$

This is the exact mathematical result we need in later work. But to go back to intuitive approximation, we may write down the formula ¹

$$g_{ij}(P') = \eta_{ij} + \frac{\kappa}{2\pi} \int (T_{ij} - \tfrac{1}{2}g_{ij}T)_{\text{ret}} r^{-1} dx^1 dx^2 dx^3, \quad (31)$$

as an approximation to the metric tensor due to the energy tensor T_{ij} ; the integration is over all space and 'ret' means the retarded value — mathematically we could equally well use the advanced value, but physicists who read a causal meaning into the formula would naturally prefer the retarded value.

¹ Cf. EINSTEIN [1916b], PAULI [1958, p. 173].

§ 3. A STATICAL EINSTEIN FIELD WITH EMBEDDED BODIES

The Newtonian astronomer surveys the universe with a powerful intuition gained more by the exercise of playful fantasy than by the consideration of technical problems. To appreciate the gravitational pull exerted by the sun on the earth he may imagine both bodies at rest, held apart by an enormous strut. To appreciate the magnitudes involved, let us make some calculations.

Using the masses stated in iv-(137) and taking the distance of sun from earth to be $1.494 \times 10^{13} \text{ cm} = 4.986 \times 10^2 \text{ sec}$, we find

$$\text{gravitational force between sun and earth} = 2.927 \times 10^{-22}. \quad (32)$$

If this thrust were maintained by a column with section equal to the cross-section of the earth, i.e. with radius $2.125 \times 10^{-2} \text{ sec}$, the pressure in this column would be

$$p = 2.063 \times 10^{-19} \text{ sec}^{-2}. \quad (33)$$

This is about three thousand atmospheres, for the pressure of the atmosphere is approximately a bar, where

$$1 \text{ bar} = 10^6 \text{ dynes cm}^{-2} = 7.423 \times 10^{-23} \text{ sec}^{-2}. \quad (34)$$

Note that the smallness of the force in (32) is physically significant, since the quantity is dimensionless. It would be foolish to say that the pressure in (33) is large or small — it all depends on what we compare it with.

Passing to relativity, it is interesting to construct a model in which bodies are held at rest relative to one another by means of a material medium which fills the space between them. We have the machinery to construct such a model without any approximation, and we shall do so. By making exact calculations ¹, we remove the argument from the realm of possible controversy and obtain formulae which may be useful in connection with problems of more direct physical appeal.

For purposes of geometrical description, we regard the coordinates x^i as rectangular coordinates in flat space-time with metric η_{ij} . We choose a function $f(x^1, x^2, x^3)$ which is arbitrary except for conditions of smoothness and of vanishing at infinity. Actually, as regards smoothness, it would suffice to take f piecewise continuous, but it is

¹ The work may be regarded as a precise treatment (without the use of isothermal coordinates) of the approximation given by CHAZY [1930, p. 153].

simpler to assume first a higher degree of smoothness and treat the case of discontinuities by a limit at the end of the argument.

Define A_{ij} by

$$A_{44} = f, \quad (35)$$

all other components vanishing. Obviously (20) is satisfied. We have

$$A = \eta^{ij}A_{ij} = -A_{44} = -f, \quad (36)$$

and the formula (30) gives

$$\gamma_{\alpha\beta}(P') = \delta_{\alpha\beta}\phi, \quad \gamma_{\alpha 4}(P') = 0, \quad \gamma_{44}(P') = \phi, \quad (37)$$

where

$$\phi = 2 \int f d\omega, \quad (38)$$

with integration over the past sheet of the null cone with vertex P' .

We may write equivalently

$$\phi = 2 \int r^{-1} f dv, \quad (39)$$

where $dv = dx^1 dx^2 dx^3$, the integration is taken over $x^4 = x^4(P')$, and r is spatial distance from P' , as shown in Fig. 2.

Pursuing the feed-back method described in IV-§ 6, we now write

$$g_{ij} = \eta_{ij} + \gamma_{ij} = \eta_{ij} + \delta_{ij}\phi, \quad (40)$$

Fig. 2 – The potential integral and investigate the properties of a universe with this metric, calculating in particular the energy tensor from the formula

$$T_{ij} = -\kappa^{-1}G_{ij}, \quad \kappa = 8\pi. \quad (41)$$

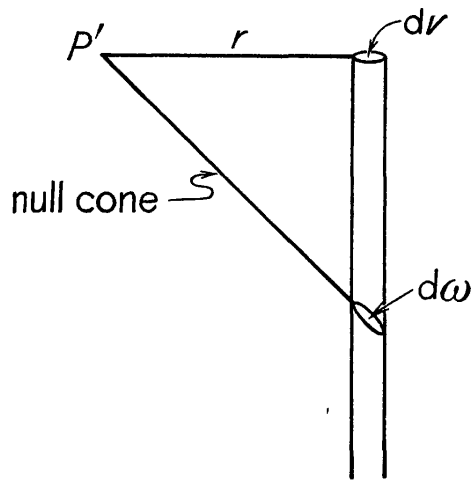
For the following calculations, we may forget that ϕ is of the form (38) or (39). In fact, we are concerned with a metric form which may be written

$$\Phi = (1 + \phi)(dx^2 + dy^2 + dz^2) - (1 - \phi)dt^2, \quad (42)$$

where ϕ is any function of (x, y, z) ; but, to have the correct signature, we understand that

$$-1 < \phi < 1. \quad (43)$$

It is not at present assumed that ϕ is small; that assumption will be made only at (53).



Denoting partial derivatives of ϕ by subscripts, we have

$$\begin{aligned}
 g_{\alpha\beta} &= \delta_{\alpha\beta}(1 + \phi), & g_{\alpha 4} &= 0, & g_{44} &= -(1 - \phi), \\
 g^{\alpha\beta} &= \frac{\delta_{\alpha\beta}}{1 + \phi}, & g^{\alpha 4} &= 0, & g^{44} &= -\frac{1}{1 - \phi}, \\
 \Gamma_{\alpha\beta}^{\gamma} &= \frac{1}{2(1 + \phi)} (\delta_{\alpha\gamma}\phi_{\beta} + \delta_{\beta\gamma}\phi_{\alpha} - \delta_{\alpha\beta}\phi_{\gamma}), \\
 \Gamma_{\alpha 4}^4 &= -\frac{\phi_{\alpha}}{2(1 - \phi)}, & \Gamma_{44}^{\alpha} &= -\frac{\phi_{\alpha}}{2(1 + \phi)},
 \end{aligned} \tag{44}$$

the other Γ 's vanishing (in all these calculations, the presence of a single 4 destroys a quantity).

The Riemann tensor is important because it is the gravitational field; from 1-(88) we find

$$\begin{aligned}
 R_{\alpha\beta\gamma\delta} &= -\frac{1}{2}(\delta_{\beta\delta}\phi_{\alpha\gamma} + \delta_{\alpha\gamma}\phi_{\beta\delta} - \delta_{\alpha\delta}\phi_{\beta\gamma} - \delta_{\beta\gamma}\phi_{\alpha\delta}) \\
 &\quad + \frac{3}{4(1 + \phi)} (\delta_{\beta\delta}\phi_{\alpha}\phi_{\gamma} + \delta_{\alpha\gamma}\phi_{\beta}\phi_{\delta} - \delta_{\alpha\delta}\phi_{\beta}\phi_{\gamma} - \delta_{\beta\gamma}\phi_{\alpha}\phi_{\delta}) \\
 &\quad - \frac{1}{4(1 + \phi)} \phi_{\rho}\phi_{\rho}(\delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma}), \\
 R_{4\beta\gamma 4} &= \frac{1}{2}\phi_{\beta\gamma} - \frac{1 - 3\phi}{4(1 - \phi^2)} \phi_{\beta}\phi_{\gamma} + \frac{1}{4(1 + \phi)} \delta_{\beta\gamma}\phi_{\rho}\phi_{\rho}.
 \end{aligned} \tag{45}$$

For the Ricci tensor and curvature invariant we get, writing $\phi_{\rho\rho} = \Delta\phi$,

$$\begin{aligned}
 R_{\alpha\beta} &= \frac{1}{2(1 + \phi)} \delta_{\alpha\beta}\Delta\phi - \frac{\phi}{1 - \phi^2} \phi_{\alpha\beta} \\
 &\quad - \frac{1 - 2\phi + 3\phi^2}{2(1 - \phi^2)^2} \phi_{\alpha}\phi_{\beta} - \frac{1}{2(1 + \phi)(1 - \phi^2)} \delta_{\alpha\beta}\phi_{\rho}\phi_{\rho}, \\
 R_{\alpha 4} &= 0, \\
 R_{44} &= \frac{\Delta\phi}{2(1 + \phi)} + \frac{\phi_{\rho}\phi_{\rho}}{2(1 + \phi)(1 - \phi^2)}, \\
 R &= \frac{1 - 3\phi}{(1 + \phi)(1 - \phi^2)} \Delta\phi - \frac{5 - 4\phi + 3\phi^2}{2(1 + \phi)(1 - \phi^2)^2} \phi_{\rho}\phi_{\rho}.
 \end{aligned} \tag{46}$$

Hence the Einstein tensor is

$$\begin{aligned}
 G_{\alpha\beta} &= \frac{\phi}{1 - \phi^2} (\delta_{\alpha\beta} \Delta\phi - \phi_{\alpha\beta}) \\
 &\quad - \frac{1 - 2\phi + 3\phi^2}{2(1 - \phi^2)^2} \phi_\alpha \phi_\beta + \frac{3 - 2\phi + 3\phi^2}{4(1 - \phi^2)^2} \delta_{\alpha\beta} \phi_\rho \phi_\rho, \\
 G_{\alpha 4} &= 0, \\
 G_{44} &= \frac{1 - \phi}{(1 + \phi)^2} \Delta\phi - \frac{3}{4} \frac{1 - \phi}{(1 + \phi)^3} \phi_\rho \phi_\rho.
 \end{aligned} \tag{47}$$

The energy tensor is then given by (41), and the principal directions and eigenvalues by

$$T_{ij} \lambda^j = \theta g_{ij} \lambda^j, \tag{48}$$

as in iv-(140). It is evident that for three of these principal vectors we have $\lambda^4 = 0$ (since $G_{\alpha 4} = 0$), the principal pressures being given by the corresponding values of θ . As for the density μ , we have $\theta = -\mu$, and

$$\begin{aligned}
 \mu &= -T_{44}/g_{44} = \kappa^{-1} g^{44} G_{44} \\
 &= -\frac{1}{8\pi} \frac{1}{(1 + \phi)^2} \left(\Delta\phi - \frac{3}{4} \frac{\phi_\rho \phi_\rho}{1 + \phi} \right).
 \end{aligned} \tag{49}$$

This completes our calculations based on the metric (42).

We now turn to (39) by which

$$\phi = 2 \int r^{-1} f dv, \quad \Delta\phi = -8\pi f. \tag{50}$$

Note that $\frac{1}{2}\phi$ is the Newtonian potential for 'density' f ; this 'density' f is not to be confused with the true density μ of the constructed universe, which is connected with f by

$$\mu = \frac{f}{(1 + \phi)^2} + \frac{3}{32\pi} \frac{\phi_\rho \phi_\rho}{(1 + \phi)^3}. \tag{51}$$

The above formulae hold in the case where f has surfaces of discontinuity, for the junction condition is the continuity of $G_{\alpha\beta} n^\beta$ and this is satisfied because, if we integrate over any closed surface in Euclidean 3-space, we have

$$\int (\delta_{\alpha\beta} \Delta\phi - \phi_{\alpha\beta}) n^\beta dS = 0. \tag{52}$$

We are now in a position to study a relativistic model in which bodies are held at rest in an embedding medium. We put $f = 0$ throughout the exterior region. But, as the formulae are somewhat complicated, we shall now approximate, treating ϕ and its derivatives as small and retaining only terms of the second order. Then (47) and (51) simplify to

$$\begin{aligned} G_{\alpha\beta} &= \phi(\delta_{\alpha\beta}\Delta\phi - \phi_{\alpha\beta}) - \tfrac{1}{2}\phi_{\alpha}\phi_{\beta} + \tfrac{3}{4}\delta_{\alpha\beta}\phi_{\rho}\phi_{\rho}, \\ G_{\alpha 4} &= 0, \\ G_{44} &= (1 - 3\phi)\Delta\phi - \tfrac{3}{4}\phi_{\rho}\phi_{\rho}, \\ \mu &= f(1 + \phi)^{-2} + \frac{3}{32\pi}\phi_{\rho}\phi_{\rho}, \end{aligned} \quad (53)$$

so that the ratio μ/f is nearly equal to unity. There are discontinuities at the boundaries of the bodies, since $\Delta\phi = 0$ outside.

To investigate the density (μ_e) of the embedding medium just outside a body of density μ_i , we have approximately

$$\frac{\mu_e}{\mu_i} = \frac{3}{32\pi} \frac{\phi_{\rho}\phi_{\rho}}{\mu_i}. \quad (54)$$

Defining the mass of a body by

$$m = \int \mu dv \sim \int f dv, \quad (55)$$

the value of ϕ at an exterior point for two spheres of masses m, m' is

$$\phi = \frac{2m}{r} + \frac{2m'}{r'} = 2V, \quad (56)$$

where r, r' are the distances from the centres and V is the Newtonian potential. (Note the factor 2; in relativity the 'magic number' associated with a body is $2m/r$, not m/r as in Newtonian gravitation.) To evaluate the ratio (54) in such a case as that presented by the sun and the earth, we note that, at the surface of either body, the value of $\phi_{\rho}\phi_{\rho}$ is very nearly that due to that body alone. Thus

$$\frac{\mu_e}{\mu_i} = \frac{3}{32\pi} \left(\frac{2m}{r^2} \right)^2 \frac{4\pi r^3}{3m} = \frac{1}{2} \frac{m}{r}, \quad (57)$$

where m is the mass of the body and r its radius. Inserting the appropriate numerical values, we obtain the following ratios of exterior

density (that of the embedding medium) to interior density:

$$\begin{aligned} &\text{for the sun: } 1.061 \times 10^{-6}, \\ &\text{for the earth: } 3.480 \times 10^{-10}, \\ &\text{for the moon: } 1.563 \times 10^{-11}. \end{aligned} \quad (58)$$

Thus the density of the fictitious embedding medium is very small compared with ordinary densities.

For the stress $S_{\alpha\beta}$, inside or outside bodies of any shape, we have by (53)

$$8\pi S_{\alpha\beta} = G_{\alpha\beta} = \phi(\delta_{\alpha\beta}\Delta\phi - \phi_{\alpha\beta}) - \frac{1}{2}\phi_{\alpha}\phi_{\beta} + \frac{3}{4}\delta_{\alpha\beta}\phi_{\rho}\phi_{\rho}. \quad (59)$$

Hence

$$8\pi S_{\alpha\beta,\beta} = \frac{1}{2}\phi_{\alpha}\Delta\phi. \quad (60)$$

This simple result leads to a striking connection between our relativistic model and the Newtonian one. For we may compute the resultant 'force' on a body by integrating the traction over its surface due to stress, and for this 'force' we get

$$\begin{aligned} \int S_{\alpha\beta}n^{\beta}d\sigma &= \int S_{\alpha\beta,\beta}dv = \frac{1}{16\pi} \int \phi_{\alpha}\Delta\phi dv \\ &= -\frac{1}{2} \int \phi_{\alpha}f dv = - \int \mu V_{\alpha} dv, \end{aligned} \quad (61)$$

where V is the Newtonian potential. In Newtonian theory this is precisely the force which must be exerted on the body in order to hold it at rest against the pull of gravity due to other bodies.

We note that the stress falls to zero at infinity like r^{-4} . For the sun-earth combination, held apart by the embedding medium, we can calculate from (59) the mean normal pressure over the surface of the earth; it comes out to be

$$\bar{p} = 5.103 \times 10^{-15} \text{ sec}^{-2}. \quad (62)$$

This is much greater than the pressure (33), as is to be expected, since the force (32) arises from the variation of the stress over the surface of the earth.

Since we have, in this section, wandered rather far from the stern tasks imposed by physical reality, it will be well to sum up and point some morals:

- (i) The general theory of relativity is bedevilled by the complexity

of its formulae. It is therefore useful to have available the formulae for R_{ijkl} and G_{ij} for the metric form (42) ¹.

- (ii) Simple exact Einstein fields are rare and it is good to have before us an example, even if its physical counterpart is not to be found in nature.
- (iii) There is no uniqueness about the universe which we have discussed, except the uniqueness of simplicity. It is the simplest universe obtainable from the combination of the g -method and the T -method described in IV-§ 6.
- (iv) The retarded-potential formula, or linear approximation, shown in (31) is so attractively simple that there is a temptation to use it uncritically; by making it the first step in an exact calculation, we are able better to appreciate its value and its limitations. Rejection of quadratic terms is dangerous. If we threw them away, we would have $S_{\alpha\beta} = 0$ in (59), and would see bodies in a state of mutual rest without any stress in the intervening medium; that would be a denial of the most elementary facts of gravitational attraction.

§ 4. TWO LEMMAS

As a preliminary to the discussion of the Cauchy problem in the next section, we shall establish two lemmas ².

Let W_{ij} be any *symmetric* tensor field in space-time with metric tensor g_{ij} . We define the *conjugate* tensor field by ³

$$W_{ij}^* = W_{ij} - \frac{1}{2}g_{ij}W, \quad W = g^{ab}W_{ab}. \quad (63)$$

Since $g_{ij}g^{ij} = 4$, we have then

$$W^* = -W, \quad (64)$$

and hence

$$W_{ij} = W_{ij}^* - \frac{1}{2}g_{ij}W^*, \quad (65)$$

so that the operator *star* is a square root of unity in the sense that

$$W_{ij}^{**} = W_{ij}. \quad (66)$$

¹ Later in the book, these tensors are evaluated for various metrics. Explicit formulae for G_j^i for a general orthogonal metric (g_{ij} diagonal) will be found in TOLMAN [1934b, p. 253] and McVIRTUE [1956, p. 68], but the gain in generality must be balanced against a loss in interpretability.

² Cf. LICHNEROWICZ [1955a, p. 31].

³ The star is used in VI-(35) and X-(7) for the *dual*, but there should be no risk of confusion.

Lemma I: Provided $g^{44} \neq 0$, the mixed components W_j^i may be expressed in terms of $W_{\alpha\beta}^*$ and W_k^4 in the linear form

$$W_j^i = A_j^{i\alpha\beta} W_{\alpha\beta}^* + B_j^{ik} W_k^4, \quad (67)$$

the coefficients being linear and quadratic functions of g^{ab} , divided by g^{44} . (Greek suffixes take the values 1, 2, 3.)

The proof is as follows. Our plan is to use the mixed form of (65), viz.

$$W_j^i = W^{*i}_j - \frac{1}{2} \delta_j^i W^*. \quad (68)$$

This gives

$$W_j^i = (g^{ia} \delta_j^b - \frac{1}{2} \delta_j^i g^{ab}) W_{ab}^*. \quad (69)$$

For $i = 4$,

$$W_j^4 = (g^{4a} \delta_j^b - \frac{1}{2} \delta_j^4 g^{ab}) W_{ab}^*. \quad (70)$$

For $j = \gamma$,

$$W_\gamma^4 = g^{4a} W_{a\gamma}^* = g^{4\alpha} W_{\alpha\gamma}^* + g^{44} W_{4\gamma}^*, \quad (71)$$

so that

$$W_{4\gamma}^* = -(g^{44})^{-1} (g^{4\alpha} W_{\alpha\gamma}^* - W_\gamma^4). \quad (72)$$

Putting $j = 4$ in (70), we get

$$W_4^4 = g^{4a} W_{a4}^* - \frac{1}{2} g^{ab} W_{ab}^*; \quad (73)$$

an important cancellation takes place here, so that

$$W_4^4 = -\frac{1}{2} g^{\alpha\beta} W_{\alpha\beta}^* + \frac{1}{2} g^{44} W_{44}^*, \quad (74)$$

and hence

$$W_{44}^* = (g^{44})^{-1} (g^{\alpha\beta} W_{\alpha\beta}^* + 2W_4^4). \quad (75)$$

Now (69) may be written in the form

$$W_j^i = C_j^{i\alpha\beta} W_{\alpha\beta}^* + C_j^{i\alpha} W_{4\alpha}^* + C_j^i W_{44}^*, \quad (76)$$

and when we substitute in this from (72) and (75), we get an expression of the form (67), and so establish the lemma. The condition $g^{44} \neq 0$ is obviously required in (72) and (75).

Lemma II: Let S_4 be a domain of space-time with $g^{44} \neq 0$, and let S_3 be the 3-space with equation $x^4 = 0$. Then the three following statements are mathematically equivalent:

- (A) $W_{ij} = 0$ in S_4 .
- (B) $W_{\alpha\beta}^* = 0$ and $W_j^4 = 0$ in S_4 .
- (C) $W_{\alpha\beta}^* = 0$ and $W_{j|i}^i = 0$ in S_4 , with $W_j^4 = 0$ in S_3 .

To prove this lemma, we note first that obviously $(A) \Rightarrow (B)$. By Lemma I, $(B) \Rightarrow (A)$. Therefore (A) and (B) are equivalent. Obviously $(B) \Rightarrow (C)$. It remains only to prove that $(C) \Rightarrow (B)$.

We assume (C) . Then, by Lemma I, and the first condition in (C) ,

$$W_j^i = B_j^{ik} W_k^4. \quad (77)$$

By the second condition in (C) we have

$$W_{j|i}^i = W_{j,4}^4 + W_{j,\alpha}^\alpha + \Gamma_{ia}^i W_j^a - \Gamma_{ij}^a W_a^i = 0. \quad (78)$$

By (77) this may be written in the form

$$W_{j,4}^4 = E_j^{k\alpha} W_{k,\alpha}^4 + F_j^k W_k^4, \quad (79)$$

the coefficients being functions of the metric tensor and its derivatives. In view of the last condition in (C) , a fundamental theorem in partial differential equations tells us that $W_j^4 = 0$, not only in S_3 , but in S_4 . Thus $(C) \Rightarrow (B)$, and the lemma is proved.

§ 5. THE CAUCHY PROBLEM IN NORMAL GAUSSIAN COORDINATES

Only in very rare circumstances can we hope to obtain solutions of the field equations which are explicit, exact, and physically significant. Therefore we have recourse to solutions in series in powers of one of the coordinates x^4 , a procedure intimately connected with existence theorems for solutions in terms of data on a 3-space $x^4 = 0$. Here we have the Cauchy problem ¹.

In the field equations

$$G_{ij} = -\kappa T_{ij}, \quad \kappa = 8\pi, \quad (80)$$

we have 10 equations connecting the 20 quantities g_{ij} , T_{ij} . The conservation equations

$$T_{ij|j}^j = 0 \quad (81)$$

are consequences of (80), not independent equations. To get a determinate mathematical problem, 10 of the 20 quantities should be assigned throughout space-time, and the other 10 quantities sought to satisfy (80). In IV-§ 6, we discussed the g -method and the T -method. In the former, g_{ij} are assigned and T_{ij} calculated by mere differentiations, but we are very likely in this way to get negative densities,

¹ Cf. LICHNEROWICZ [1955a], PHAM MAU QUAN [1953b], [1955b], FOURÈS-BRUHAT [1948b], [1950], [1952], [1955], [1956].

and also tensions where, physically, we would prefer pressures. The T -method is more promising, but from the standpoint of the Cauchy problem, it seems best to mix the two methods. In fact, what we shall do in the following treatment of the Cauchy problem is to employ normal Gaussian coordinates, so that, as in I-(213), we have

$$g_{\alpha 4} = 0, \quad g_{44} = -1, \quad (82)$$

taking the parametric lines of x^4 to be timelike, and assign $T_{\alpha\beta}$ throughout space-time. Thus 10 quantities are assigned, and for the other 10 ($g_{\alpha\beta}$, T_{i4}) we have to solve the 10 equations (80). This choice is indicated by the structure of the differential system relative to the Cauchy problem, the initial data being taken on $x^4 = 0$.

Let us, for clarity, list the quantities as follows:

$$\text{assigned in space-time: } g_{i4}, T_{\alpha\beta} \quad (83)$$

$$\text{unknowns: } g_{\alpha\beta}, T_{i4}. \quad (84)$$

Let us define W_{ij} by ¹

$$W_{ij} = G_{ij} + \kappa T_{ij}. \quad (85)$$

Then the conjugate, as in (65), is

$$W_{ij}^* = R_{ij} + \kappa T_{ij}^*, \quad (86)$$

where

$$T_{ij}^* = T_{ij} - \frac{1}{2}g_{ij}T, \quad T = g^{ab}T_{ab}, \quad (87)$$

because

$$G_{ij}^* = R_{ij}, \quad R_{ij}^* = G_{ij}. \quad (88)$$

The field equations (80) now read

$$W_{ij} = 0. \quad (89)$$

Noting that (82) implies

$$g^{\alpha 4} = 0, \quad g^{44} = -1, \quad (90)$$

we now appeal to Lemma II of § 4. It tells us that the equations (89) are equivalent to

$$W_{\alpha\beta}^* = 0, \quad W_{j|i}^i = 0, \quad (91)$$

¹ The following general argument might be carried through with inclusion of the cosmological constant Λ of IV-(108); we would then write

$$W_{ij} = G_{ij} - \Lambda g_{ij} + \kappa T_{ij}. \quad (85a)$$

with the condition

$$W_i^4 = 0 \text{ for } x^4 = 0. \quad (92)$$

These are the same as

$$R_{\alpha\beta} + \kappa T_{\alpha\beta}^* = 0, \quad T_{j|i}^i = 0, \quad (93)$$

with the *consistency conditions*

$$G_{4i} + \kappa T_{4i} = 0 \text{ for } x^4 = 0. \quad (94)$$

To attack the Cauchy problem for the system (93), we note that by I-(217) with $\varepsilon = -1$ we may write the first of (93) in the form

$$g_{\alpha\beta,44} = 2\bar{R}_{\alpha\beta} - \frac{1}{2}A g_{\alpha\beta,4} + g^{\mu\nu} g_{\alpha\mu,4} g_{\beta\nu,4} + 2\kappa T_{\alpha\beta}^*, \quad (95)$$

where $\bar{R}_{\alpha\beta}$ is the intrinsic Ricci tensor of $x^4 = 0$ and

$$A = g^{\mu\nu} g_{\mu\nu,4}, \quad (96)$$

while the second of (93) gives

$$T_{4j,4} = -T_{j,4}^4 = T_{j,\alpha}^\alpha + \Gamma_{ai}^i T_j^a - \Gamma_{ij}^a T_a^i. \quad (97)$$

We now assign as Cauchy data on $x^4 = 0$ the values of the 16 quantities

$$g_{\alpha\beta}, \quad g_{\alpha\beta,4}, \quad T_{i4}, \quad (98)$$

remembering however that they must be chosen to satisfy the four conditions (94), which by I-(219) may be written

$$\begin{aligned} A_{,\alpha} - D^\sigma g_{\alpha\sigma,4} + 2\kappa T_{\alpha 4} &= 0, \\ \bar{R} - \frac{1}{4}A^2 + \frac{1}{4}B + 2\kappa T_{44} &= 0, \end{aligned} \quad (99)$$

for $x^4 = 0$, where \bar{R} is the intrinsic curvature invariant of $x^4 = 0$ and

$$B = g^{\mu\nu} g^{\rho\sigma} g_{\mu\rho,4} g_{\nu\sigma,4}. \quad (100)$$

Since (95) and (97) give explicitly the values of

$$g_{\alpha\beta,44}, \quad T_{i4,4}, \quad (101)$$

the derivatives of the Cauchy data, in terms of the Cauchy data themselves, we know that a solution exists in the neighbourhood of $x^4 = 0$, provided the Cauchy data are chosen to satisfy (99) — this is most important physically, for it is in (99) that we come up against

the physical condition of positive density IV-(146a), viz.

$$T_{44} > 0. \quad (102)$$

If we have satisfied (99), the solution reads

$$\begin{aligned} g_{\alpha\beta} &= (g_{\alpha\beta})_0 + x^4(g_{\alpha\beta,4})_0 + \tfrac{1}{2}(x^4)^2(g_{\alpha\beta,44})_0 + \dots, \\ T_{4j} &= (T_{4j})_0 + x^4(T_{4j,4})_0 + \dots \end{aligned} \quad (103)$$

where the coefficients are either Cauchy data or are expressible in terms of Cauchy data by (95) and (97). Should we desire higher terms in the expansions, we would get them in terms of the Cauchy data by differentiating (95) and (97) and substituting in the results the values given by the equations themselves.

It should be clearly understood that the way in which we split the problem in (83) and (84) was merely a mathematical device, without physical motivation. The argument applies to any Einstein field, i.e. to any set of 20 quantities g_{ij} , T_{ij} satisfying the field equations (80), subject of course to conditions of smoothness which we have not complicated the discussion by mentioning, since they are best discussed in those special cases where lack of smoothness intrudes itself.

The preceding treatment of the Cauchy problem applies in particular to fields in vacuo, for which the field equations read

$$G_{ij} = 0; \quad (104)$$

we have merely to delete the terms in T_{ij} from the work. Thus, in normal Gaussian coordinates, the equations (95) become

$$g_{\alpha\beta,44} = 2\bar{R}_{\alpha\beta} - \tfrac{1}{2}A g_{\alpha\beta,4} + g^{\mu\nu} g_{\alpha\mu,4} g_{\beta\nu,4}. \quad (105)$$

The Cauchy data are now $g_{\alpha\beta}$, $g_{\alpha\beta,4}$ on $x^4 = 0$, and they must be chosen to satisfy (99) with $T_{ij} = 0$. If we write

$$\psi_{\alpha\beta} = g_{\alpha\beta,4} \quad \text{for } x^4 = 0, \quad (106)$$

then the consistency conditions (99) may be written

$$\begin{aligned} \psi_{\beta||\alpha}^\beta &= \psi_{\alpha||\beta}^\beta, \\ (\psi_\beta^\beta)^2 - \psi_\beta^\alpha \psi_\alpha^\beta &= 4\bar{R}. \end{aligned} \quad (107)$$

These are equations in the 3-space $x^4 = 0$, so that there are only three independent variables x^α ; the metric tensor is $g_{\alpha\beta}$ and subscripts

are raised by $g^{\alpha\beta}$; the double stroke indicates covariant differentiation, using the Christoffel symbols

$$\Gamma_{\beta\gamma}^{\alpha} = g^{\alpha\rho}[\beta\gamma, \rho]. \quad (108)$$

The equations (107) must be regarded as of great importance because they contain, as it were, the general theory of *gravitational waves*¹. Once they have been solved for the 12 quantities $(g_{\alpha\beta}, \psi_{\alpha\beta})$, the field can be developed from (105) in the neighbourhood of $x^4 = 0$. But we should hardly speak of *solving* (107), because they are highly redundant — only 4 equations for 12 unknowns. It looks as if it would be easy to satisfy them but in fact it is not.

There are two ways of simplifying drastically the consistency conditions (107). The first plan² is to put $\psi_{\alpha\beta} = 0$, so that we have only to satisfy $\bar{R} = 0$. If $g'_{\alpha\beta}$ is a subsidiary metric in $x^4 = 0$, related conformally to $g_{\alpha\beta}$ by $g_{\alpha\beta} = \phi^4 g'_{\alpha\beta}$, the consistency condition becomes³

$$\Delta'\phi + \frac{1}{8}\bar{R}'\phi = 0, \quad (109)$$

where Δ' is the tensorial Laplace operator, and \bar{R}' the curvature invariant, both calculated for the metric tensor $g'_{\alpha\beta}$.

The second plan is to assume that $x^4 = 0$ is intrinsically flat, so that $\bar{R} = 0$ and we may use rectangular Cartesian coordinates x^α . The consistency conditions (107) now read

$$\psi_{\beta\beta,\alpha} = \psi_{\alpha\beta,\beta}, \quad (\psi_{\beta\beta})^2 = \psi_{\alpha\beta}\psi_{\alpha\beta}. \quad (110)$$

There is still redundancy — only 4 equations for 6 unknowns. But if we now take $\psi_{\alpha\beta}$ to be of the form $\psi_{\alpha\beta} = u_\alpha u_\beta$, where u_α is some vector field, the redundancy disappears since the last of (110) is identically satisfied and we are left with the following three equations for u_α :

$$(u_\beta u_\beta)_{,\alpha} = u_\alpha u_{\beta,\beta} + u_{\alpha,\beta} u_\beta. \quad (111)$$

Since we are now in a Euclidean 3-space, it is useful to think of these equations hydrodynamically, regarding u_α as the velocity of a fluid in steady motion, $u_\beta u_\beta$ being the square of the speed and $u_{\alpha,\beta} u_\beta$ the acceleration. We can satisfy (111) by giving the fluid a velocity in the

¹ See Chap. IX.

² This plan gives *time-symmetric* solutions in the sense that the series (103) for $g_{\alpha\beta}$ contains only even powers of x^4 ; cf. FOURÈS-BRUHAT [1956], WEBER and WHEELER [1957], BRILL [1959a, b], ARAKI [1959].

³ Note the *fourth* power of ϕ . The calculations are similar to those following VIII-(36), but in three dimensions, not four.

direction of the x^3 -axis, combined with a whirling about that axis. We put

$$u_1 = -\omega(r)x^2, \quad u_2 = \omega(r)x^1, \quad u_3 = u_3(r), \quad r^2 = (x^1)^2 + (x^2)^2, \quad (112)$$

where ω is any smooth function which vanishes sufficiently rapidly at infinity. Then from (111) we get the single equation

$$\frac{d}{dr}(\omega^2 r^2 + u_3^2) = -\omega^2 r, \quad (113)$$

and this is satisfied by taking

$$u_3^2 = C - \omega^2 r^2 - \int_0^r \omega^2 r dr, \quad (114)$$

the constant C being chosen to make u_3^2 positive for all values of r . Having thus obtained u_α to satisfy (111) and hence $\psi_{\alpha\beta}$ to satisfy (109), we have the requisite initial values for use with the field equations (105). We must not of course attempt to simplify these equations by inserting values appropriate only to $x^4 = 0$; but we note that the initial value is

$$(g_{\alpha\beta,44})_0 = -\frac{1}{2}\psi_{\mu\mu}\psi_{\alpha\beta} + \psi_{\alpha\mu}\psi_{\beta\mu} = \frac{1}{2}u_\alpha u_\beta u_\mu u_\mu. \quad (115)$$

From the form of (112), the gravitational waves discussed above may be described as cylindrical¹. These waves are of a very special type; a close study of the general consistency conditions (107) might lead to other more interesting types of wave.

§ 6. THE CAUCHY PROBLEM IN NORMAL GAUSSIAN COORDINATES FOR A PERFECT FLUID

In the preceding section we dealt with the Cauchy problem for a medium with energy tensor T_{ij} , and it might be thought that the argument there given would apply in particular to the case of a perfect fluid, for which, as in IV-(84),

$$T_{ij} = (\mu + p)V_i V_j + p g_{ij}. \quad (116)$$

But it does not apply. The essential feature of (116) is that T_{ij} is degenerate, with three equal eigenvalues. In the preceding procedure, we assigned $T_{\alpha\beta}$ arbitrarily in space-time and then found T_{i4} through the field equations. There is no reason to suppose that, in such a procedure, the degeneracy of T_{ij} is preserved. In fact, the preceding

¹ Cf. Chap. IX-§ 3.

method is no good at all for the case of a perfect fluid, and we have to make a fresh attack, taking from the preceding work only those parts that are applicable to the case of the perfect fluid. It is well worth doing this, because the problem is indeed the basic problem of celestial mechanics. In Newtonian mechanics we think of the elements of the solar system given with respect to position and velocity at $t = 0$, and the motion of the system unfolds by the solution of certain differential equations, ordinary if the bodies are rigid, partial if the bodies are fluid. As already mentioned, the concept of rigidity does not pass over into relativity, and we have little option but to treat the bodies as fluid. It is best not to trouble at first about concentrations of matter, but rather to work with a general fluid field.

As before, we shall use normal Gaussian coordinates, so that as in (82) and (90)

$$g_{\alpha 4} = 0, \quad g_{44} = -1, \quad g^{\alpha 4} = 0, \quad g^{44} = -1. \quad (117)$$

The field equations are

$$G_{ij} + \kappa T_{ij} = 0, \quad \kappa = 8\pi, \quad (118)$$

with T_{ij} as in (116), and we have also the equation

$$V_i V^i = -1. \quad (119)$$

We see here 11 equations for the 12 quantities

$$g_{\alpha\beta}, \quad V_i, \quad \mu, \quad \phi, \quad (120)$$

and so we are one equation short.

There are four fairly reasonable ways of supplying an additional equation. First, we might put

$$\phi = 0, \quad (121)$$

so that the matter becomes incoherent (a dust cloud); we have already considered this in IV-§ 4, and seen that then the stream-lines are geodesics, but that is far from telling the whole story. Secondly; we might assume a density-pressure relationship

$$f(\mu, \phi) = 0, \quad (122)$$

which of course contains (121) as a very special case. Thirdly, we might assume

$$(\mu V^i)_{|i} = 0, \quad (123)$$

an equation which might be described as conservation of mass. By IV-(87) this implies $V^i_{|i} = 0$, so that the motion is without expansion.

Fourthly, we might modify (123) to read

$$(\rho V^i)_{|i} = 0, \quad (124)$$

where ρ is defined by

$$\rho = \mu + p. \quad (125)$$

By iv-(86) this implies that p is constant along each stream-line.

The smallness of p/μ in physical situations suggests that the difference between (123) and (124) is trivial, but the remarks following iv-(98) warn us against such rash conclusions. If our interest lies in incompressible fluids, then (123) is the natural equation to adopt. However, (124) is more convenient mathematically, and we shall adopt it here, admitting arbitrariness in the choice.

We now have before us the 12 equations (118), (119) and (124) for the 12 quantities (120). However, it is best to regard (119) as giving V^4 :

$$V^4 = (1 + V_\alpha V^\alpha)^{\frac{1}{2}}. \quad (126)$$

So now we have the 11 unknowns

$$g_{\alpha\beta}, \quad V_\alpha, \quad \rho, \quad p, \quad (127)$$

and they are to satisfy the 11 equations (118) and (124). As in (93) and (94), (118) is equivalent to

$$R_{\alpha\beta} + \kappa T_{\alpha\beta}^* = 0, \quad T_{j|i}^i = 0, \quad (128)$$

with the consistency conditions

$$G_{4i} + \kappa T_{4i} = 0. \quad (129)$$

Noting that

$$\begin{aligned} T &= 3p - \mu = 4p - \rho, \\ T_{ij}^* &= \rho V_i V_j + (\tfrac{1}{2}\rho - p)g_{ij}, \end{aligned} \quad (130)$$

the first of (128) becomes, as in (95),

$$\begin{aligned} g_{\alpha\beta,44} &= 2\bar{R}_{\alpha\beta} - \tfrac{1}{2}A g_{\alpha\beta,4} + g^{\mu\nu} g_{\alpha\mu,4} g_{\beta\nu,4} \\ &\quad + 2\kappa\rho V_\alpha V_\beta + \kappa(\rho - 2p)g_{\alpha\beta}, \end{aligned} \quad (131)$$

and the second of (128) gives

$$\rho V^i V_{j|i} + p_{,j} = 0, \quad (132)$$

while the consistency conditions (129) read, as in (99),

$$\begin{aligned} A_{,\alpha} - D^\sigma g_{\alpha\sigma,4} + 2\kappa\rho V_\alpha V_4 &= 0, \\ \bar{R} - \tfrac{1}{4}A^2 + \tfrac{1}{4}B + 2\kappa\rho V_4^2 - 2\kappa p &= 0. \end{aligned} \quad (133)$$

As Cauchy data on $x^4 = 0$ we assign the values of

$$g_{\alpha\beta}, \quad g_{\alpha\beta,4}, \quad V_\alpha, \quad \rho, \quad p, \quad (134)$$

chosen subject to (133), and we seek to solve for the quantities

$$g_{\alpha\beta,44}, \quad V_{\alpha,4}, \quad \rho_{,4}, \quad p_{,4}. \quad (135)$$

Now $g_{\alpha\beta,44}$ are given by (131), and for the other five quantities in (135) we have the five equations contained in (124) and (132). Putting $j = 4$ in (132), we get

$$p_{,4} = -\rho V^i V_{4|i}, \quad (136)$$

which gives $p_{,4}$ once $V_{\alpha,4}$ have been found. Putting $j = \alpha$ in (132), we get

$$V^4 V_{\alpha,4} + V^\beta V_{\alpha,\beta} - \Gamma_{i\alpha}^k V^i V_k + \rho^{-1} p_{,\alpha} = 0, \quad (137)$$

and this gives $V_{\alpha,4}$. Finally we turn to (124) and get

$$\rho_{,i} V^i + \rho V^i_{|i} = 0, \quad (138)$$

or

$$\rho_{,4} V^4 + \rho_{,\alpha} V^\alpha + \rho V^i_{|i} = 0, \quad (139)$$

and this gives $\rho_{,4}$. The quantities (135) are thus given in terms of the Cauchy data (134) by (131), (137), (139) and (136).

But we have still to consider the consistency conditions (133). In the vacuum case, we had in (107) 4 consistency conditions to be satisfied by 12 quantities; now in (133) we have 4 consistency conditions to be satisfied by the 17 quantities (134). Let us exhibit these conditions in a better form by writing, as in (106),

$$\psi_{\alpha\beta} = g_{\alpha\beta,4}; \quad (140)$$

they then read

$$\begin{aligned} \psi_{\beta,\alpha}^\beta - \psi_{\alpha||\beta}^\beta + 2\kappa\rho V_\alpha V_4 &= 0, \\ \bar{R} - \frac{1}{4}(\psi_\beta^\beta)^2 + \frac{1}{4}\psi_\beta^\alpha \psi_\alpha^\beta + 2\kappa\rho V_4^2 - 2\kappa p &= 0. \end{aligned} \quad (141)$$

Obviously with so many redundant quantities there can be no unique procedure for solving these equations. In seeking a plan of attack, we may try to get a solution corresponding to a linear approximation. Accordingly in a tentative spirit we put

$$g_{\alpha\beta} = \delta_{\alpha\beta} + \gamma_{\alpha\beta}, \quad (142)$$

and treat $\gamma_{\alpha\beta}$ as small; further, we treat $\psi_{\alpha\beta}$ and V_α as small, so that we write $V_4 = -1$. Then (141) reduce to

$$\psi_{\beta\beta,\alpha} - \psi_{\alpha\beta,\beta} - 2\kappa\rho V_\alpha = 0, \quad (143)$$

$$\Delta\gamma_{\alpha\alpha} - \gamma_{\alpha\beta,\alpha\beta} + 2\kappa\mu = 0, \quad (144)$$

where Δ is the Euclidean Laplace operator. We have separated $\psi_{\alpha\beta}$ and $\gamma_{\alpha\beta}$. Putting

$$\psi_{\alpha\beta} = 2(v_{\alpha,\beta} + v_{\beta,\alpha}) - \delta_{\alpha\beta}v_{\gamma,\gamma}, \quad (145)$$

we satisfy (143) by taking (remember $\kappa = 8\pi$)

$$v_\alpha = \int \frac{\rho V_\alpha}{r} dv, \quad (146)$$

and putting

$$\gamma_{\alpha\beta} = 2\delta_{\alpha\beta}\chi, \quad (147)$$

we satisfy (144) by taking

$$\chi = \int \frac{\mu dv}{r}. \quad (148)$$

In (146) and (148), r is distance (in Euclidean metric) between the volume element dv and the point at which v_α or χ is computed. In this way we obtain solutions of the linearized consistency conditions (143), (144), with V_α , ρ , p chosen arbitrarily.

It would not be wise to attach much importance to this result in itself. We should use it only as a basis for obtaining exact solutions of the exact consistency conditions (141), and that we shall now do. But since, as already mentioned, we are attacking the central problem of celestial mechanics, it will be well to restate the case.

We have four coordinates x^i with x^4 timelike. In terms of the Cauchy data (134), the equations (131), (137), (139), (136), with their derivatives with respect to x^4 , yield the coefficients in the power series

$$\begin{aligned} g_{\alpha\beta} &= (g_{\alpha\beta})_0 + x^4(g_{\alpha\beta,4})_0 + \frac{1}{2}(x^4)^2(g_{\alpha\beta,44})_0 + \dots, \\ V_\alpha &= (V_\alpha)_0 + x^4(V_{\alpha,4})_0 + \frac{1}{2}(x^4)^2(V_{\alpha,44})_0 + \dots, \\ \rho &= (\rho)_0 + x^4(\rho_{,4})_0 + \frac{1}{2}(x^4)^2(\rho_{,44})_0 + \dots, \\ p &= (p)_0 + x^4(p_{,4})_0 + \frac{1}{2}(x^4)^2(p_{,44})_0 + \dots \end{aligned} \quad (149)$$

But we cannot use these power series until we have solved (141). To the creator of universes (cf. iv-§ 6) this presents no problem: he can

choose $\psi_{\alpha\beta}$, $g_{\alpha\beta}$ and p arbitrarily and solve for ρ and V_α , remembering (126). But he has no assurance that ρ will come out positive, and universes created in this way without guidance are likely to be quite unnatural. The realist starts with a Newtonian picture of the solar system (if that is what he is interested in) and chooses reasonable pilot-values of V_α , ρ and p . Using these pilot-values, he calculates $\psi_{\alpha\beta}$ and $g_{\alpha\beta}$ from (142) and (145)–(148). Then, using the feed-back method of IV-§ 6, he substitutes these values of $\psi_{\alpha\beta}$ and $g_{\alpha\beta}$ in (141), and also the pilot-value of p . He obtains an exact solution of (141) by solving for V_α and ρ . Since p is always under control, it can be made zero outside the sun and the planets, and the Cauchy data can fail to be realistic only through the existence of a density outside the sun and the planets, which density might come out negative. A rough numerical calculation shows that this artificial density has its greatest absolute value near the sun, where it is about $10^{-14} \text{ sec}^{-2}$ or $10^{-7} \text{ g cm}^{-3}$.

For what range of x^4 can we expect the power series (149) to be valid? We are using normal Gaussian coordinates, constructed by drawing geodesics normal to $x^4 = 0$. These coordinates fail and render (149) invalid as soon as two adjacent geodesics intersect. The question of intersection can be treated by geodesic deviation, but we arrive more quickly at an estimate by using Newtonian ideas, treating the geodesics as the histories of particles which start from rest at time $x^4 = 0$ and fall freely through matter, without resistance. In a homogeneous sphere of density ρ , a particle, starting from rest in any position, falls in to the centre in time

$$x^4 = \sqrt{\frac{3\pi}{16\rho}}. \quad (150)$$

If we accept 100 g cm^{-3} as the density at the centre of the sun, this is certainly the greatest density in the solar system, and if we insert this value in (150) we get a rough upper bound for the valid range of x^4 in the power series (149), when applied to the solar system. The range comes out about 300 sec. Before this time has elapsed, one must select a new 3-space to act as basis for new Gaussian coordinates.

§ 7. CHARACTERISTICS AND SHOCK WAVES

The word *characteristic* is so overworked in mathematics that it is liable to cause confusion of thought, even in the domain with which

we are here concerned, viz. partial differential equations in space-time.

Let us first deal briefly with characteristic curves. Consider a partial differential equation of the first order,

$$F(x, y) = 0, \quad (151)$$

to be satisfied by a function $S(x)$; here y stands for the partial derivatives

$$y_i = S_{,i}. \quad (152)$$

The *characteristic curves* of (151) are those curves which satisfy the differential equations

$$\frac{dx^i}{du} = \frac{\partial F}{\partial y_i}, \quad \frac{dy_i}{du} = -\frac{\partial F}{\partial x^i}. \quad (153)$$

The importance of these equations lies in the fact that, by solving a set

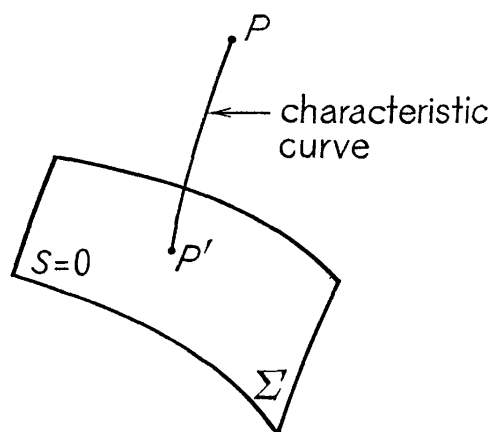


Fig. 3 – Characteristic curves

of *ordinary* differential equations, we can obtain all solutions of the *partial* differential equation (151). This is done as follows. We seek the solution S of (151) such that $S = 0$ on some 3-space Σ (Fig. 3). At each point of Σ we choose quantities y_i such that (151) is satisfied, and we draw the congruence of characteristic curves out from Σ , treating y_i as a set of four quantities, without regard to (152). Taking any point $P(x)$,

we draw through it a characteristic curve of this congruence; let it meet Σ at P' . Then the equation

$$S(x) = \int_{P'}^P y_i dx^i, \quad (154)$$

with integration along the characteristic curve, defines $S(x)$ as a function of position. It is easy to see that $S(x)$ satisfies the partial differential equation (151) and the condition $S = 0$ on Σ . The argument is that already given at I-(180) and need not be repeated here. Also, as in I-§ 7, we can generalize the condition on Σ , giving arbitrary values to S instead of the value zero.

In particular, if the partial differential equation is

$$F(x, y) = \frac{1}{2}g^{ij}y_i y_j = 0, \quad (155)$$

the equations for the characteristic curves are

$$\frac{dx^i}{du} = g^{ij}y_j, \quad \frac{dy_i}{du} = -\frac{1}{2}g^{jk}_{,i}y_j y_k, \quad (156)$$

and from them we obtain

$$\frac{\delta}{\delta u} \frac{dx^i}{du} = 0, \quad g_{ij} \frac{dx^i}{du} \frac{dx^j}{du} = 0. \quad (157)$$

These are the equations of a null geodesic and (155) is the equation of a null surface (cf. I-§ 7). In fact, *the characteristic curves associated with null surfaces are null geodesics*. In stating this, we are really repeating what has been said in I-§ 7.

To sum up, characteristic curves are curves associated with a partial differential equation of the *first* order, and, once we have found those curves, we have all solutions of the partial differential equation.

We pass now to a different use of the word *characteristic*, and to avoid confusion we shall introduce it by reference to shock waves. We sometimes have occasion to think of quantities which are continuous but which have discontinuous derivatives. For example, in Newtonian attractions, the potential and its first derivatives are continuous across the surface of a sphere of matter, but there are discontinuities in the second derivatives (Poisson's equation inside, Laplace's outside). In space-time we may likewise have continuous quantities with derivatives discontinuous across a 3-space Σ . Following the terminology of hydrodynamics, we say then that Σ is a *shock wave*; the equivalent mathematical term is *characteristic*, but we shall not use it.

We shall discuss shock waves in space-time relative to scalar partial differential equations of the second order and also relative to the vacuum field equations. Electromagnetic shock waves will be treated in x-§ 2. It may be remarked that the type of argument used below is a modern equivalent of an old dodge in optics, viz, the passage from 'physical optics' to 'geometrical optics' by considering periodic waves of high frequency.

Consider a partial differential equation of the second order,

$$F = 0, \quad (158)$$

where F is an invariant involving an invariant S and its partial derivatives $S_{,i}$, $S_{,ij}$. A shock wave Σ is a 3-space across which S and $S_{,i}$ are continuous, but there are some discontinuities in the second derivatives $S_{,ij}$. To investigate shock waves, we change to new coordinates \bar{x}^i such that Σ has the equation $\bar{x}^4 = 0$. Then the following quantities are continuous across Σ :

$$\bar{S}, \quad \bar{S}_{,\alpha}, \quad \bar{S}_{,4}, \quad \bar{S}_{,\alpha\beta}, \quad \bar{S}_{,\alpha 4}. \quad (159)$$

Thus the only second derivative which can be discontinuous is $\bar{S}_{,44}$.

We now look on the equation (158) as an algebraic equation for $\bar{S}_{,44}$. If it possesses a unique solution, then $\bar{S}_{,44}$ is continuous across Σ . Thus a shock wave has a negative definition: Σ is a shock wave if $F = 0$ does not give a unique value for $\bar{S}_{,44}$.

As an example, consider, in a given space-time, the *generalized wave equation*

$$g^{ij}S_{,ij} = 0. \quad (160)$$

Since the argument actually involves only the second derivatives, it applies equally to a more general equation of the form

$$g^{ij}S_{,ij} + B = 0, \quad (161)$$

where B involves S and $S_{,i}$. But let us for simplicity think of (160). In coordinates \bar{x}^i for which Σ is $\bar{x}^4 = 0$, it reads

$$\bar{g}^{44}\bar{S}_{,44} + \dots = 0, \quad (162)$$

the dots standing for terms involving the quantities (159). Obviously the condition for a shock wave is

$$\bar{g}^{44} = 0, \quad (163)$$

since, if this does not hold, we get $\bar{S}_{,44}$ uniquely from (162). We now pass back to general coordinates x^i , writing

$$\bar{x}^4 = f(x), \quad (164)$$

so that Σ is $f(x) = 0$. We have

$$\bar{g}^{44} = g^{ij}f_{,i}f_{,j}, \quad (165)$$

and so we see that *the shock waves for the wave equation (160), or more generally for (161), are the 3-spaces $f(x) = \text{const.}$ where f satisfies*

$$g^{ij}f_{,i}f_{,j} = 0. \quad (166)$$

These are null surfaces and the associated characteristic curves are null geodesics. Since the null surfaces are themselves (in mathematical language) the characteristic surfaces of the wave equation, one says that the *bicharacteristics* of the wave equation are null geodesics.

Let us now consider *gravitational shock waves* associated with the vacuum field equations

$$G_{ij} = 0, \quad (167)$$

which are equivalent to

$$R_{ij} = 0. \quad (168)$$

By *shock wave* here we understand a 3-space Σ across which g_{ij} and $g_{ij,k}$ are continuous (as indeed is demanded by the condition for admissible coordinates), but there are essential discontinuities in some of the second derivatives $g_{ij,km}$; by *essential* we mean that they cannot be transformed away by the use of other admissible coordinates.

We attack the problem of gravitational shock waves in the same manner as before, taking coordinates \bar{x}^i for which the equation of Σ is $\bar{x}^4 = 0$. We have then continuity of the quantities

$$\bar{g}_{ij}, \quad \bar{g}_{ij,k}, \quad \bar{g}_{ij,\alpha\beta}, \quad \bar{g}_{ij,4\alpha}, \quad (169)$$

and the only quantities in which discontinuities can occur are

$$\bar{g}_{ij,44}. \quad (170)$$

By 1-(86) and 1-(104), we find

$$\begin{aligned} \bar{R}_{\alpha\beta} &= \frac{1}{2}\bar{g}^{44}\bar{g}_{\alpha\beta,44} + \bar{F}_{\alpha\beta}, \\ \bar{R}_{\alpha 4} &= -\frac{1}{2}\bar{g}^{4\beta}\bar{g}_{\alpha\beta,44} + \bar{F}_{\alpha 4}, \\ \bar{R}_{44} &= \frac{1}{2}\bar{g}^{\alpha\beta}\bar{g}_{\alpha\beta,44} + \bar{F}_{44}, \end{aligned} \quad (171)$$

where the F 's involve the quantities (169).

We see at once that the field equations

$$\bar{R}_{ij} = 0 \quad (172)$$

do not determine all the second derivatives (170) uniquely, because four of them ($g_{i4,44}$) do not occur in (171) at all. Thus discontinuities in second derivatives may occur across any 3-space. However, we must remember the word *essential* in the definition of shock waves. It is possible, without going outside the class of admissible coordinates,

to eliminate certain discontinuities in second derivatives, and the best plan is to use Gaussian coordinates \bar{x}^i (in general, skew) such that, as above, Σ has the equation $\bar{x}^4 = 0$ and we have, as in 1-(212),

$$\bar{g}_{\alpha 4,4} = 0, \quad \bar{g}_{44} = \varepsilon = \pm 1. \quad (173)$$

Then $\bar{g}_{i4,44} = 0$, and out of the ten second derivatives (170) we have only to consider the six $\bar{g}_{\alpha\beta,44}$. Then it is clear that, unless

$$\bar{g}^{44} = 0, \quad (174)$$

the field equations (172) determine these six derivatives uniquely, whereas, if (174) holds, they do not. Hence (174) is the equation which defines gravitational shock waves. But (174) is the same as (163), and we conclude that *gravitational shock waves are null surfaces*¹ *with the equation* (166).

In ordinary parlance, we may say that 'gravitational shock waves travel with the speed of light', and we may think of the bicharacteristics (null geodesics) as 'gravitational rays'. But of course such statements must be taken cum grano salis. However, there is little doubt that if, at some time, we should think seriously of quantized elements of gravity (gravitons) as we now think of quantized elements of light (photons), then it would be natural to ascribe to the gravitons null geodesics for world-lines.

¹ Cf. LEVI-CIVITA [1930], PASTORI [1939b], B. FINZI [1949], LICHNEROWICZ [1955a, p. 33].

CHAPTER VI

INTEGRAL CONSERVATION LAWS AND EQUATIONS OF MOTION

§ 1. THE CONCEPT OF INTEGRAL CONSERVATION LAWS

Consider, in Newtonian mechanics, a closed system; by this we mean a system which is not subject to any external forces. Then, if ρ is the density and u_α the velocity, we have the law of *conservation of linear momentum* in the form

$$\int \rho u_\alpha d_3v = \text{const.}, \quad (1)$$

and the law of *conservation of angular momentum* in the form

$$\int \rho (x_\alpha u_\beta - x_\beta u_\alpha) d_3v = \text{const.}, \quad (2)$$

where ‘const.’ means ‘independent of time’. There is also a conservation law for energy, simple in the case of a conservative system, but passing outside mechanics proper if mechanical energy is dissipated into heat.

In attempting to construct the relativistic analogue of these integral conservation laws of Newtonian mechanics, the first and most obvious difficulty lies in the fact that there is no unique invariant way of slicing up space-time into a single infinity of spacelike sections. It is advisable to take at once a wider view, better suited to relativity, and regard an integral conservation law as a statement of the vanishing of an integral taken over a *closed* 3-space in space-time. To state this more precisely, let V_3 be a closed 3-space (Fig. 1), with an enclosed 4-dimensional region V_4 . Let N^i be the unit normal to V_3 , drawn

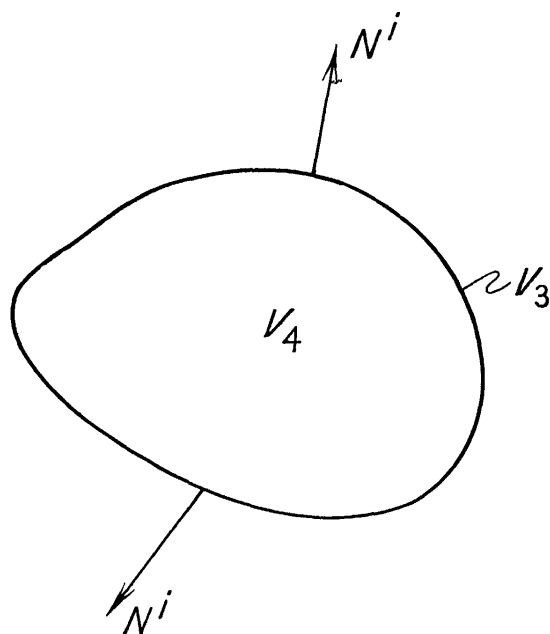


Fig. 1 – General scheme for a conservation law

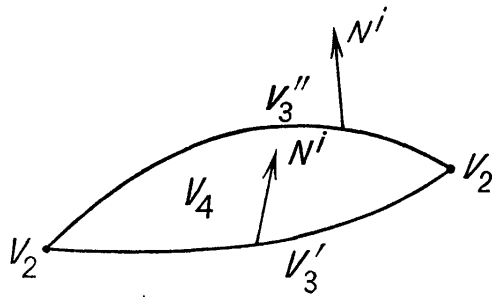


Fig. 2 – Conservation law in open form

($V_3 = V_3' + V_3''$), both spanning the same closed V_2 (Fig. 2). If we continue to denote by N^i the outward normal on V_3'' , but change the meaning of N^i on V_3' so that it becomes the inward normal, then (3) may be written

$$\int_{V_3'} \varepsilon(N) F_i N^i d_3v = \int_{V_3''} \varepsilon(N) F_i N^i d_3v. \quad (4)$$

Either of these integrals may be called a *flux*, and the fact that the flux for V_3' is the same as that for V_3'' is rightly referred to by the word *conservation*. Obviously the flux is the same for all open 3-spaces spanning the same closed V_2 , but in general it depends on the choice of V_2 . With due caution, we may let V_2 recede to infinity.

In the discussion of integral conservation laws we may also appeal to the generalized Stokes' theorems of 1-§ 10, taking a somewhat wider view than above.

Let V_1 be a closed curve in space-time and V_2 an open 2-space spanning it. Let F_i be an *arbitrary* vector field. By 1-(244)

$$\oint_{V_1} F_i dx^i = \int_{V_2} F_{i|j} d\tau^{ij}. \quad (5)$$

Then the integral on the right has the same value for all 2-spaces spanning V_1 , and this may be said to give an integral conservation law. Since this is a double integral, not a triple integral as in (4), this law resembles the classical law of Gauss in electrostatics.

Let us now increase the dimensionality, taking a closed 2-space V_2 spanned by an open V_3 . Let F_{ij} be an *arbitrary* tensor field. Then by 1-(245)

$$\oint_{V_2} F_{ij} d\tau^{ij} = \int_{V_3} F_{ij|k} d\tau^{ijk}. \quad (6)$$

¹ The symbols d_3v and d_4v denote invariant elements of 3-volume and 4-volume, and the integrands occurring in §§ 1-4 are invariant under transformation of coordinates in the domain of integration. In §§ 5-7 the integrands are not invariant.

outwards. Then if there exists a vector field F_i such that

$$\oint_{V_3} \varepsilon(N) F_i N^i d_3v = 0 \quad (3)$$

for every closed V_3 , we call (3) an *integral conservation law*¹.

The meaning of the word *conservation* is made clear if we split the closed V_3 into two open parts

It is not necessary that F_{ij} should have any particular symmetry, but only its skew-symmetric part contributes in (6). Clearly the integral on the right has the same value for all 3-spaces spanning a given closed V_2 , and so we have a conservation law rather like (4). But there is an important difference: in (4) the vector F_i was not arbitrary — it had to satisfy (3) —, but in (6) the tensor F_{ij} is arbitrary. In fact, (6) may be regarded as a factory for making integral conservation laws: our task is to select an F_{ij} to yield a law of physical interest.

It is sometimes necessary or desirable to confine our operations to a portion of space-time, say V_4 . Under this restriction, the use of (5) or (6) requires care, because the spanning of V_1 in the former case, and of V_2 in the latter, might possibly carry us out of V_4 . As an illustration, Fig. 3 shows a tube T in space-time which it is forbidden to enter. The

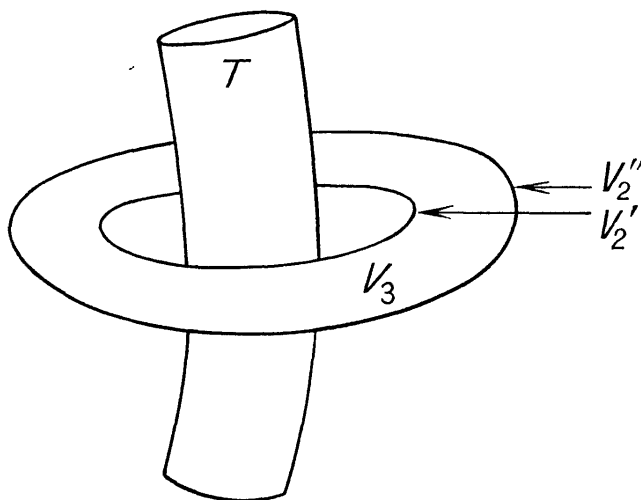


Fig. 3 – Conservation outside a forbidden region T

closed 2-space V_2' cannot be spanned without entering T , and so we cannot apply (6) to V_2' . But the closed V_2 formed of V_2' and V_2'' can be spanned by an open V_3 without entering T , and a conservation law of the type (6) exists.

It should be remarked that our spatial intuitions extend only to three dimensions, and a diagram like Fig. 3 which portrays space-time as if it were only of three dimensions, although useful, must be handled with some care. In all dubious cases, resort to formulae. Let (x, y, z, t) be coordinates ranging from $-\infty$ to $+\infty$ in space-time. Then

$$x^2 + y^2 + z^2 \leq a^2$$

represents a domain T of space-time, and closed 2-spaces V_2' , V_2'' are

defined by

$$\begin{aligned} V'_2: x^2 + y^2 + z^2 &= b^2, & t &= 0, \\ V''_2: x^2 + y^2 + z^2 &= c^2, & t &= 0. \end{aligned}$$

Here a, b, c are constants, and if $a < b < c$ we have a situation as in Fig. 3. The formula

$$x^2 + y^2 + z^2 \leq b^2, \quad t = 0$$

defines an open V_3 which spans V'_2 but cuts T . The formula

$$b^2 \leq x^2 + y^2 + z^2 \leq c^2, \quad t = 0$$

defines an open V_3 which spans the V_2 formed out of V'_2 and V''_2 without cutting T . Another V_3 which does this is given by

$$t = (x^2 + y^2 + z^2 - b^2)(c^2 - x^2 - y^2 - z^2), \quad t \geq 0.$$

§ 2. INTEGRAL CONSERVATION LAWS BASED ON THE EINSTEIN TENSOR

As in IV-(111), we have the field equations

$$G_{ij} = -\kappa T_{ij}, \quad \kappa = 8\pi. \quad (7)$$

The left hand side is geometrical, the right hand side mechanical. Since the two sides are equal, it makes no mathematical difference which we work with. But it does make a considerable psychological difference, because geometry is a domain from which semantic controversies are happily absent, and as long as we stick to geometry we do not become involved in such metaphysical questions as the meaning of the word *energy*, for example. We shall therefore, in the interests of peace of mind, work with G_{ij} and develop results which are geometrically true — the physical meanings will then follow from the connection (7) between G_{ij} and T_{ij} .

In the present section we are concerned primarily with only one property of G_{ij} , the identity

$$G^i{}_{|j} = 0, \quad (8)$$

which is in fact the *differential* conservation law. To pass to integral form, we introduce an arbitrary vector field λ_i and integrate over any portion V_4 of space-time, obtaining

$$\int_{V_4} G^i{}_{|j} \lambda_i d_4v = 0. \quad (9)$$

To turn this into an integral conservation law of the form (3), we integrate by parts and use Green's theorem I-(257), obtaining

$$\oint_{V_3} \varepsilon(N) G^{ij} \lambda_i N_j d_3 v = \int_{V_4} G^{ij} \lambda_{i|j} d_4 v, \quad (10)$$

where V_3 is the closed 3-space bounding V_4 and N^i is its unit outward normal.

If only the right hand side of (10) were to vanish, we would have an integral conservation law in the form

$$\oint_{V_3} \varepsilon(N) G^{ij} \lambda_i N_j d_3 v = 0. \quad (11)$$

To obtain this result, we may choose λ_i to satisfy the equation

$$G^{ij} \lambda_{i|j} = 0, \quad (12)$$

and this can be done in a variety of ways, since we have only one equation for four unknowns. We may in fact choose an arbitrary vector field v_i and put

$$\lambda_i = \psi v_i, \quad (13)$$

where ψ is some scalar. Then (12) is satisfied if ψ satisfies the partial differential equation

$$G^{ij} v_i \psi_{,j} + G^{ij} v_{i|j} \psi = 0. \quad (14)$$

The corresponding integral conservation law reads

$$\oint_{V_3} \varepsilon(N) \psi G^{ij} v_i N_j d_3 v = 0. \quad (15)$$

This plan may be made more systematic by choosing v_i to be any one of the four unit eigenvectors of G_{ij} , so that

$$G^{ij} v_i = \phi v^j, \quad (16)$$

where ϕ is the corresponding eigenvalue. Then the partial differential equation (14) for ψ becomes

$$\phi v^j \psi_{,j} + (\phi_{,j} v^j + \phi v^j_{|j}) \psi = 0. \quad (17)$$

This equation can be interpreted in terms of the geometry of the congruence of curves having v^i for unit tangent. Measuring s from some 3-space which cuts the v -lines, and denoting by $\Delta\sigma$ the 3-volume of the normal section of a thin tube of v -lines, we have

$$v^j \psi_{,j} = \frac{d\psi}{ds}, \quad v^j \phi_{,j} = \frac{d\phi}{ds}, \quad v^j_{|j} = \frac{1}{\Delta\sigma} \frac{d\Delta\sigma}{ds}, \quad (18)$$

and (17) becomes

$$\phi \frac{d\psi}{ds} + \left(\frac{d\phi}{ds} + \phi \frac{1}{\Delta\sigma} \frac{d\Delta\sigma}{ds} \right) \psi = 0, \quad (19)$$

which gives

$$\psi\phi\Delta\sigma = \Delta k, \quad (20)$$

where Δk is constant along each ν -line. The conservation law (15) then reads

$$\oint_{V_3} \varepsilon(N) \frac{\Delta k}{\Delta\sigma} \nu^j N_j d_3v = 0. \quad (21)$$

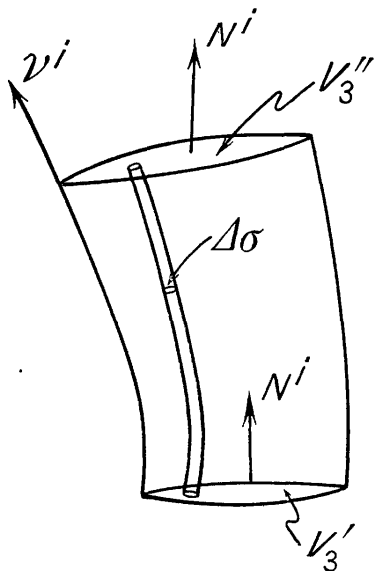


Fig. 4.—Tube of eigenlines of G_{ij}

Let us now take for V_4 a tube in space-time formed from the eigen ν -lines and terminated by 3-dimensional caps V_3' , V_3'' (Fig. 4). Then $\nu^j N_j = 0$ on the sides of the tube, and (21) may be written

$$\begin{aligned} \int_{V_3'} \varepsilon(N) \frac{\Delta k}{\Delta\sigma} \nu^j N_j d_3v \\ = \int_{V_3''} \varepsilon(N) \frac{\Delta k}{\Delta\sigma} \nu^j N_j d_3v, \end{aligned} \quad (22)$$

the normal N^i having the sense shown in Fig. 4. Now $\varepsilon(N)\nu^j N_j$ has one sign over V_3' and V_3'' , and by the projection formula we have

$$|\varepsilon(N)\nu^j N_j| d_3v = \Delta\sigma. \quad (23)$$

Hence the conservation law (21) may be exhibited in the very simple form

$$\int_{V_3'} \psi\phi\Delta\sigma = \int_{V_3''} \psi\phi\Delta\sigma. \quad (24)$$

§ 3. SPACE-TIME ADMITTING A GROUP OF MOTIONS

We have seen how the vector field λ_i may be chosen so as to make the right hand side of (10) vanish, the integral conservation law (11) resulting. This may be done much more directly if space-time admits a group of motions.

The concept of a group of motions may be described as follows¹. Consider in space-time (Fig. 5) a congruence of curves (C), each curve having on it an assigned parameter u and hence a definite tangent vector

$$\xi^i = \frac{dx^i}{du}. \quad (25)$$

Take any two events, P_1 and P'_1 , and let the values of the parameter at these events be u_1 and u'_1 . Now displace these two events along C to positions P_2, P'_2 with parameters u_2, u'_2 , where

$$u_2 = u_1 + \Delta u, \quad u'_2 = u'_1 + \Delta u. \quad (26)$$

In terms of the world-function², space-time is said to *admit a group of motions* if

$$\Omega(P_2 P'_2) = \Omega(P_1 P'_1), \quad (27)$$

provided this holds for every pair of selected events P_1, P'_1 and for every value of the increment Δu .

It is obvious that (27) is equivalent to

$$\Omega_i \xi^i + \Omega_{i'} \xi^{i'} = 0, \quad (28)$$

where the left hand side is a 2-point invariant for an arbitrary pair of events P, P' . If v is a special parameter on the geodesic $P'P$ and $V^i = dx^i/dv$ (so that $\delta V^i/\delta v = 0$), then, by II-(17), (28) is equivalent to

$$V^i \xi_i - V^{i'} \xi_{i'} = 0. \quad (29)$$

Let us define Ψ by

$$\Psi = V^i \xi_i - V^{i'} \xi_{i'}, \quad (30)$$

where ξ_i stands for any vector field. Differentiating along the geodesic $P'P$, we get

$$\frac{d\Psi}{dv} = V^i \xi_{i|j} V^j. \quad (31)$$

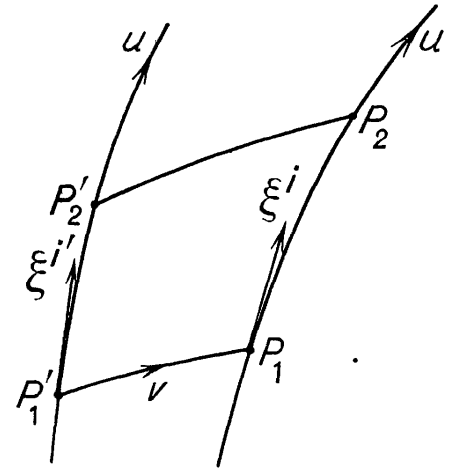


Fig. 5—Space-time admitting a group of motions

¹ See EISENHART [1926] for a more complete treatment.

² The definition of a group of motions may also be presented in infinitesimal form, the events P_1, P'_1 being adjacent and the increment Δu being infinitesimal.

Since, obviously, $\Psi = 0$ when $P' = P$, it follows that the condition, necessary and sufficient, that ξ_i should define a group of motions is

$$\xi_{i|j} + \xi_{j|i} = 0. \quad (32)$$

These ten equations are called the *equations of Killing*. We shall call ξ_i the *Killing vector* and $\xi_{i|j}$ the *Killing tensor*; note that this tensor is skew-symmetric.

Here follow some properties of the Killing vector and tensor. By I-(94),

$$\xi_{i|jk} - \xi_{i|kj} = R_{ajk}\xi^a, \quad (33)$$

and, applying two cyclic permutations to ijk , adding, and using (32) and I-(90), we get

$$\xi_{i|jk} + \xi_{j|ki} + \xi_{k|ij} = 0; \quad (34)$$

we say that the Killing tensor has a *vanishing cyclic divergence*. If we introduce the dual by

$$\xi^{*ij} = \frac{1}{2}\eta^{ijkm}\xi_{k|m}, \quad (35)$$

we have

$$\xi^{*ij}{}_{|j} = 0. \quad (36)$$

If we apply (32) to the first term in (34), we get

$$-\xi_{j|ik} + \xi_{j|ki} + \xi_{k|ij} = 0; \quad (37)$$

the first two terms may be combined as in (33), and hence the second-order covariant derivative of the Killing vector can be expressed in terms of the Riemann tensor and the Killing vector itself as follows:

$$\xi_{k|ij} = R_{ajik}\xi^a = R_{kija}\xi^a. \quad (38)$$

If we multiply this by g^{ij} , we have (in terms of the generalized d'Alembertian)

$$\square \xi_k = g^{ij}\xi_{k|ij} = R_{ka}\xi^a.$$

Since the 3-dimensional element of extension $d\tau^{ijk}$ is skew-symmetric in each pair of indices, it follows from (34) that

$$\xi_{i|jk}d\tau^{ijk} = 0; \quad (40)$$

hence, by (6), for any closed V_2 we have

$$\oint_{V_2} \xi_{i|j}d\tau^{ij} = 0. \quad (41)$$

If we insert ξ_i for λ_i in (10), the right hand side vanishes by (32), and so, for space-time admitting a group of motions with Killing vector ξ_i , we have the integral conservation law

$$\oint_{V_3} \varepsilon(N) G^{ij} \xi_i N_j d_3v = 0. \quad (42)$$

§ 4. INTEGRAL CONSERVATION LAWS BASED ON THE RIEMANN TENSOR

The conserved integrals in (11) and (42) receive no contributions from those parts of V_3 which lie in empty space-time, since there $G_{ij} = 0$. We would expect, however, that the gravitational field even in empty space-time should make its contribution, and so we turn to integral conservation laws based, not on G_{ij} , but on R_{ijkm} .

If F^{ij} is any skew-symmetric tensor, then, as in (6),

$$\oint_{V_2} F^{ij} R_{ijkm} d\tau^{km} = \int_{V_3} F^{ij}{}_{|n} R_{ijkm} d\tau^{kmn}; \quad (43)$$

the term arising from differentiating the Riemann tensor has disappeared, since, by the Bianchi identity I-(98),

$$R_{ijkm|n} d\tau^{kmn} = 0. \quad (44)$$

Hence, no matter what F^{ij} may be, we have the integral conservation law

$$\oint_{V_3} F^{ij}{}_{|n} R_{ijkm} d\tau^{kmn} = 0. \quad (45)$$

Our task is to pick out some F^{ij} which is defined by the geometry of space-time and which yields a law which has some resemblance to Newtonian laws of conservation.

We might, for example, take for F^{ij} an eigentensor of the Riemann tensor, by which we mean one which satisfies equations of the form

$$F^{ij} R_{ijkm} = \phi F_{km}. \quad (46)$$

However, such a choice does not appear interesting in the present connection, and we shall not pursue it further.

Instead, we shall make use of the world-function $\Omega(PP')$, where P is the current event of integration and P' some *base event*, held fixed during the argument. We define

$$F_{a'}^{ij} = -\frac{1}{4} \kappa^{-1} \eta^{ijpq} \Omega_{,p} \Omega_{,a'q}, \quad \kappa = 8\pi. \quad (47)$$

This is a skew-symmetric tensor relative to transformations of co-

ordinates at P and a covariant vector with respect to transformations at P' . Taking any closed V_2 , let us define $M_{a'}$ by

$$M_{a'} = \oint_{V_2} F_{a'}^{ij} R_{ijkl} d\tau^{kl}. \quad (48)$$

This is a covariant vector for transformations at P' . The presence of the primed subscript does not interfere with (43) and so

$$M_{a'} = \int_{V_3} F_{a'|n}^{ij} R_{ijkl} d\tau^{kl}, \quad (49)$$

where V_3 is any open 3-space spanning V_2 . This is a well defined geometrical quantity, and we need to give it a name. But that is a dangerous procedure, because the adoption of a suggestive name is likely to involve confusion with other uses of that name. For reasons to be discussed later, we shall call $M_{a'}$ the *flux of total 4-momentum across the open 3-space V_3 , relative to the base event P'* .

Likewise we define

$$F_{a'b'}^{ij} = \frac{1}{4} \kappa^{-1} \eta^{ijpq} (\Omega_{a'p} \Omega_{b'q} - \Omega_{b'p} \Omega_{a'q} - \Omega \Omega_{a'p} \Omega_{b'q}). \quad (50)$$

For transformations at P' this is a skew-symmetric tensor. We define

$$H_{a'b'} = \oint_{V_2} F_{a'b'}^{ij} R_{ijkl} d\tau^{kl}, \quad (51)$$

and so we have

$$H_{a'b'} = \int_{V_3} F_{a'b'|n}^{ij} R_{ijkl} d\tau^{kl}. \quad (52)$$

We call $H_{a'b'}$ the *flux of total angular momentum across the open 3-space V_3 , relative to the base event P'* .

Whatever names we give to them, the fact is that $M_{a'}$ and $H_{a'b'}$ have values independent of the particular open V_3 which spans any given closed V_2 . Equivalently, we may state that, for any *closed* V_3 , we have

$$M_{a'} = 0, \quad H_{a'b'} = 0. \quad (53)$$

As regards the presence of the base event P' , we certainly expect (by Newtonian analogy) that a base event should appear in a definition of angular momentum. It may seem out of place in the case of 4-momentum, but we must remember that the existence of the Newtonian law of conservation of linear momentum is closely connected with the fact that Euclidean space admits translations, and this is not so for Riemannian space-time.

The presence of the closed V_2 in the above work may seem to add an extraneous feature. But this is merely a frank acknowledgment of a possible non-convergence present in conservation theories. We might let V_2 move out to spatial infinity, and then, if the Riemann tensor tends to zero at a suitable rate, the integrals (48) and (51) may tend to finite limits, and then (49) and (52) will be finite conserved integrals over the whole of "space".

There is a possibility of using the above definitions to obtain an invariant definition of the history of the mass-centre of all the matter in the universe. This will become more realistic after we have made the approximations given below. But for the moment we remark that, given any closed V_2 , the equations

$$H_{a'b'}M^{b'} = 0, \quad (54)$$

although apparently four in number, are related by an identity on account of the skew-symmetry of H . They therefore form a set of three equations for the event P' , and so define a *history of the mass-centre*, a world-line¹. This history will in general depend on the choice of V_2 , but, under the conditions of convergence stated above, it may be given absolute significance.

We shall now put (49) and (52) into a different form, which indeed we might have adopted originally had we preferred to use Green's theorem rather than that of Stokes; but then the conservation would have been less evident. We make use of 1-(249), which gives

$$d\tau^{kmn} = \varepsilon(N)\eta^{kmnr}N_r d_3v, \quad (55)$$

where N_r is the unit normal to V_3 and d_3v an invariant element of volume. Then (49) reads

$$\begin{aligned} M_{a'} &= -\frac{1}{4}\kappa^{-1} \int_{V_3} \eta^{ijpq}(\Omega_p \Omega_{a'q})_{|n} R_{ijkm} \varepsilon(N) \eta^{kmnr} N_r d_3v \\ &= -\kappa^{-1} \int_{V_3} (\Omega_p \Omega_{a'q})_{|n} \tilde{R}^{pqnr} \varepsilon(N) N_r d_3v, \end{aligned} \quad (56)$$

where \tilde{R} is the double dual as in 1-(115). Likewise

$$\begin{aligned} H_{a'b'} &= \kappa^{-1} \int_{V_3} (\Omega_{a'p} \Omega_{b'q} - \Omega_{b'p} \Omega_{a'q} \\ &\quad - \Omega \Omega_{a'p} \Omega_{b'q})_{|n} \tilde{R}^{pqnr} \varepsilon(N) N_r d_3v. \end{aligned} \quad (57)$$

Let us now give some justification for the names assigned to $M_{a'}$

¹ Cf. SYNGE [1956a, p. 219] for the definition in flat space-time.

and $H_{a'b'}$ by making an approximate calculation for a weak field, i.e. one for which the Riemann tensor is small. For such a field we know by II-(95) that

$$\Omega_{ij} = g_{ij} + O_1, \quad \Omega_{ij'} = -g_{ij'} + O_1, \quad (58)$$

where $g_{ij'}$ is the parallel propagator and O_1 means a small quantity of the order of the Riemann tensor. All the third derivatives of Ω are O_1 [cf. II-§ 5]. In (56) $M_{a'}$ is already O_1 , and we have

$$M_{a'} = \kappa^{-1} \int_{V_3} g_{pn} g_{a'q} \tilde{R}^{pqnr} \varepsilon(N) N_r d_3v + O_2, \quad (59)$$

and so, by I-(124),

$$M_{a'} = -\kappa^{-1} \int_{V_3} g_{a'q} G^{qr} \varepsilon(N) N_r d_3v + O_2. \quad (60)$$

Let $\lambda^{a'}$ be an arbitrary unit vector at P' and let λ^a be the result of parallel transport along the geodesic $P'P$, so that

$$\lambda^{a'} g_{a'q} = \lambda_q. \quad (61)$$

Multiplying (60) by $\lambda^{a'}$ we get

$$M_{a'} \lambda^{a'} = -\kappa^{-1} \int_{V_3} \lambda_q G^{qr} \varepsilon(N) N_r d_3v + O_2, \quad (62)$$

or in terms of the energy tensor ($G_{ij} = -\kappa T_{ij}$)

$$M_{a'} \lambda^{a'} = \int_{V_3} \varepsilon(N) T^{qr} \lambda_q N_r d_3v + O_2. \quad (63)$$

We are concentrating here on the principal part, displayed as an integral; but we have in Chap. II the machinery to make the calculations explicit to O_2 inclusive.

Now, turning back to IV-(26), we recognize the integral in (63): it is the flux of 4-momentum, resolved in the direction of λ^q , across the finite target V_3 . This integral does not satisfy a conservation law, whereas $M_{a'}$ does, and so the residue O_2 is important. To avoid confusion, we must distinguish between the accurate component of $M_{a'}$ and the integral in (63). If we call the former the flux of *total* 4-momentum, we may fitly call the latter the flux of *mechanical* 4-momentum. The difference (O_2) may be described as the flux of *gravitational* 4-momentum — it exists in vacuo.

Let us now carry out a similar approximation for angular momentum.

The integrand in (57) is

$$\begin{aligned}
& (g_{a'p}g_{b'n}\Omega_q - g_{a'p}\Omega_{b'g_{qn}} - g_{b'p}g_{a'n}\Omega_q + g_{b'p}\Omega_{a'g_{qn}} \\
& \quad - \Omega_n g_{a'p}g_{b'q})\tilde{R}^{pqnr}\varepsilon(N)N_r + O_2 \\
& = g_{a'p}g_{b'n}\Omega_q(\tilde{R}^{pqnr} + \tilde{R}^{pnrq} + \tilde{R}^{prqn})\varepsilon(N)N_r \\
& \quad + (\Omega_{a'g_{b'p}} - \Omega_{b'g_{a'p}})g_{qn}\tilde{R}^{pqnr}\varepsilon(N)N_r + O_2. \tag{64}
\end{aligned}$$

The first part vanishes by I-(120) and so, by I-(124), (57) gives

$$H_{a'b'} = \kappa^{-1} \int_{V_3} (\Omega_{a'g_{b'p}} - \Omega_{b'g_{a'p}}) G^{pr}\varepsilon(N) N_r d_3v + O_2. \tag{65}$$

Let $\lambda_{(c)}^{a'}$ be an orthonormal tetrad at P' and $\lambda_{(c)}^a$ the result of parallel transport along the geodesic $P'P$, so that

$$\lambda_{(c)}^{a'} g_{a'p} = \lambda_{(c)p}, \tag{66}$$

and, as in II-(150), the quasi-Cartesian coordinates of P relative to P' are

$$X_{(c)} = -\Omega_{a'}\lambda_{(c)}^{a'}. \tag{67}$$

Then (65) gives the following values for the invariant components of $H_{a'b'}$ on the tetrad:

$$\begin{aligned}
H_{(cd)} &= H_{a'b'}\lambda_{(c)}^{a'}\lambda_{(d)}^{b'} \\
&= -\kappa^{-1} \int_{V_3} (X_{(c)}\lambda_{(d)}^p - X_{(d)}\lambda_{(c)}^p) G_{pr}\varepsilon(N) N^r d_3v + O_2 \\
&= \int_{V_3} \varepsilon(N) (X_{(c)}\lambda_{(d)}^p - X_{(d)}\lambda_{(c)}^p) T_{pr} N^r d_3v + O_2. \tag{68}
\end{aligned}$$

Comparison with IV-(26) and (2) shows this integral to be an appropriate expression for the flux of *mechanical* angular momentum, and we may repeat the same sort of remarks as those made above in the case of 4-momentum, emphasizing that $H_{a'b'}$ is the flux of *total* angular momentum and that the difference (O_2) may be regarded as a flux of *gravitational* angular momentum.

So much for approximations. Let us return to the exact equations (56) and (57). In each we see an integral spread over an open V_3 with an integrand depending on a base event P' . We may write

$$\begin{aligned}
M_{a'} &= \int_{V_3} M_{a'}^r \varepsilon(N) N_r d_3v, \\
H_{a'b'} &= \int_{V_3} H_{a'b'}^r \varepsilon(N) N_r d_3v, \tag{69}
\end{aligned}$$

where

$$\begin{aligned} M_{a'}^r &= -\kappa^{-1}(\Omega_p \Omega_{a'q})|_n \tilde{R}^{pqnr}, \\ H_{a'b'}^r &= \kappa^{-1}(\Omega_{a'p} \Omega_{b'q} \Omega_q - \Omega_{b'p} \Omega_{a'q} \Omega_q - \Omega \Omega_{a'p} \Omega_{b'q})|_n \tilde{R}^{pqnr}. \end{aligned} \quad (70)$$

These quantities are 2-point tensors with the indicated character with respect to transformations at P and P' . In (70) all reference to V_3 has disappeared, and these quantities may be regarded as localized densities of 4-momentum and angular momentum, which however depend on the base event P' . We note in passing the following co-incidence limits as P' tends to P :

$$[M_{a'}^r] = -\kappa^{-1}G_{a'}^r, \quad [H_{a'b'}^r] = 0. \quad (71)$$

For a weak field we have

$$\begin{aligned} M_{a'}^r &= -\kappa^{-1}g_{a'q}G^{qr} + O_2, \\ H_{a'b'}^r &= \kappa^{-1}(\Omega_{a'g}g_{b'p} - \Omega_{b'g}g_{a'p})G^{pr} + O_2. \end{aligned} \quad (72)$$

The essential conservation property consists in the vanishing of integrals over a closed V_3 , as in (53), and this property is preserved if we take, not one base event P' , but a set of them and add together the separate $M_{a'}$ and the separate $H_{a'b'}$. This suggests that we might be able to get rid of dependence on a base event statistically by integrating with respect to P' over space-time. But we cannot integrate a vector or a tensor, and the best we can do (without introducing extraneous elements) is to write down the integrals

$$\int_{V_4} M_{a'} M^{a'} d_4 v', \quad \int_{V_4} H_{a'b'} H^{a'b'} d_4 v', \quad \int_{V_4} H_{a'b'} H^{*a'b'} d_4 v', \quad (73)$$

where the star indicates the dual,

$$H^{*a'b'} = \frac{1}{2}\eta^{a'b'c'd'} H_{c'd'}, \quad (74)$$

and the integration is in each case over a domain V_4 of space-time. These integrals depend on nothing but the closed V_2 which occurs in (48) and (51) and on the domain V_4 . As remarked earlier, we may be able to get rid of V_2 by letting it recede to spatial infinity. Then we might obtain three absolute invariants by dividing the integrals in (73) by the 4-volume of V_4 , or a power of that 4-volume, and proceeding to a limit in which V_4 embraces the whole of space-time. But the issue is dubious, without investigation of convergence in special cases, and we shall leave the matter there.

§ 5. SPACE-TIME VIEWED FROM THE EUCLIDEAN STANDPOINT

The formulae of analysis have a universal validity. They express for the most part relations between numbers. But every mathematician turns to geometry from time to time, because our powerful intuitions about the space we seem to live in throw light on complicated analytical situations. Even in elementary algebra, we appreciate the classification of the roots of a quadratic equation best by drawing a parabolic graph.

In making graphs (to use the word in a general sense) we intuitively use Euclidean space, usually of two or three dimensions. Aided by analogy, our intuition extends, perhaps a little hazily, to Euclidean space of higher dimensionality. Thus, whatever the physical properties of the universe — whether physical space exist or not —, the concept of Euclidean space is one of those things we would hate to do without.

In relativity we are concerned with *events* and each event is a number-tetrad x^i . It would be possible to develop relativity without the language of geometry, for we might regard g_{ij} from a purely analytical standpoint, and work entirely in formulae. That is not the method of this book. We have laboured to depict nature in terms of pictures in a 4-dimensional curved space-time, and there is no reason to apologize for the use of such a powerful way of looking at things. But some people do not like it. They would prefer to illuminate formulae where they require illumination by reference, not to Riemannian geometry, but to that Euclidean geometry customarily employed, as indicated above, to aid the mind when formulae become oppressive.

Let us then look at relativity from the Euclidean standpoint. In doing this, we are not implying in any remote sense that space-time is Euclidean; the Euclidean scaffolding which we put in is of our own making, and we put it in for our intellectual comfort and for that alone.

Let V_4 be a domain of space-time, or possibly the whole of space-time. Let x^i be any system of coordinates in V_4 , with a 1 : 1 correspondence between events and number-tetrads x^i . Let g_{ij} be a set of (ten) symmetric functions of the coordinates with continuous first derivatives. These functions are however subject to certain algebraic conditions. We take the first three coordinates x^α spacelike and x^4 timelike; this means that we impose on g_{ij} the conditions

$$g_{11} > 0, \quad g_{22} > 0, \quad g_{33} > 0, \quad g_{44} < 0. \quad (75)$$

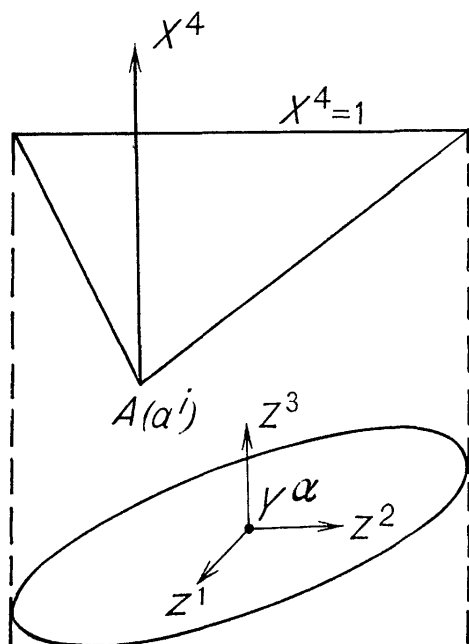


Fig. 6 – The null cone and its ellipsoidal section, viewed in the Euclidean manner

To view V_4 in the Euclidean manner, we set up a 4-dimensional Euclidean space with rectangular Cartesian coordinates x^i , a domain of this space corresponding point by point with V_4 .

Let us see, in Euclidean terms, what conditions must be satisfied by g_{ij} in order that the form $g_{ij}dx^i dx^j$ may have the correct signature. Take any point A with $x^i = a^i$, and write the coordinates x^i of a current point P in the form

$$x^i = a^i + X^i, \quad (76)$$

so that X^i are the coordinates of P relative to A . The null cone at A is tangent to the cone with equation

$$g_{ij}X^i X^j = 0, \quad (77)$$

with g_{ij} evaluated at A (Fig. 6). The section of this cone by the plane $X^4 = 1$ is the quadric surface

$$g_{\alpha\beta}X^\alpha X^\beta + 2g_{\alpha 4}X^\alpha + g_{44} = 0. \quad (78)$$

The basic requirement about the form $g_{ij}dx^i dx^j$ is essentially that (78) should represent a *real ellipsoid*. If (78) is an ellipsoid, it has a unique centre $X^\alpha = Y^\alpha$, say, where

$$g_{\alpha\beta}Y^\beta + g_{\alpha 4} = 0. \quad (79)$$

Hence

$$\det g_{\alpha\beta} \neq 0; \quad (80)$$

we may then define $\gamma^{\alpha\beta}$ by

$$\gamma^{\alpha\beta}g_{\alpha\gamma} = \delta_\gamma^\beta, \quad (81)$$

and write the solution of (79) in the form

$$Y^\alpha = -\gamma^{\alpha\beta}g_{\beta 4}. \quad (82)$$

We now transform to the centre, writing

$$Z^\alpha = X^\alpha - Y^\alpha, \quad (83)$$

and (78) becomes

$$g_{\alpha\beta}Z^\alpha Z^\beta = \gamma^{\alpha\beta}g_{\alpha 4}g_{\beta 4} - g_{44}. \quad (84)$$

If this is an ellipsoid, the quadratic form on the left must be definite, and by (75) it must be positive-definite. To sum up, given $g_{44} < 0$, the condition (necessary, and also sufficient) that (78) should be a real ellipsoid is simply

$$g_{\alpha\beta}Z^\alpha Z^\beta \text{ positive-definite,} \quad (85)$$

or, equivalently, the three roots of

$$\det(g_{\alpha\beta} - \theta\delta_{\alpha\beta}) = 0 \quad (86)$$

are all positive; note that (85), with $g_{44} < 0$, implies

$$\gamma^{\alpha\beta}g_{\alpha 4}g_{\beta 4} - g_{44} > 0. \quad (87)$$

The condition imposed on g_{ij} by signature is (85) with $g_{44} < 0$.

This elementary argument has been given at length as an illustration of the interplay of algebra with Euclidean intuitions. We have been talking about Riemannian space-time, but the invariance of the metric form under general coordinate transformation has been completely suppressed.

When we think about space-time in this Euclidean way, we are inclined to attach some importance to 'straight lines' with equations

$$x^i = ua^i + b^i, \quad (88)$$

where u is a parameter and the a 's and b 's constants. But these lines must be carefully distinguished from the geodesics, which satisfy

$$-\frac{d^2x^i}{du^2} + \Gamma_{jk}^i \frac{dx^j}{du} \frac{dx^k}{du} = 0, \quad (89)$$

where u is a special parameter.

Let us now take a look at Green's theorem from the Euclidean standpoint. The formula I-(257) is pleasing to the tensorialist, for he sees in it invariant integrands and invariant elements of volume. It applies however only to the integration of the divergence of a vector field. We might do better to go back to the elementary formula I-(233) and extend it to space-time, regarded as Euclidean in the sense explained above. Indeed, the usual method extended to Euclidean 4-space leads at once to

$$\int_{V_4} U_{,i} dx^1 dx^2 dx^3 dx^4 = \oint_{V_3} U n_i dS; \quad (90)$$

here V_4 is a domain of space-time bounded by the closed V_3 , and n_i and dS are respectively the outward unit normal and the 3-element of V_3 , both calculated for the assumed Euclidean metric. There is no reason to think of only one function U ; we have likewise

$$\int_{V_4} U^{jk},_i dx^1 dx^2 dx^3 dx^4 = \oint_{V_3} U^{jk} n_i dS. \quad (91)$$

Let the equation of V_3 be

$$f(x) = 0, \quad (92)$$

with f increasing outwards. Then

$$n_i = f_{,i} (f_{,j} f_{,j})^{-\frac{1}{2}}, \quad (93)$$

and, by orthogonal projection,

$$n_4 dS = \nu dx^1 dx^2 dx^3, \quad (94)$$

where $\nu = +1$ or -1 according as n_4 is positive or negative. Thus (91) may be written

$$\int_{V_4} U^{jk},_i dx^1 dx^2 dx^3 dx^4 = \oint_{V_3} U^{jk} \nu (f_{,i}/f_{,4}) dx^1 dx^2 dx^3. \quad (95)$$

If $U^{jk} n_k$ vanishes on V_3 , then $U^{jk} f_{,k} = 0$, and (95) with $i = k$ gives

$$\int_{V_4} U^{jk},_k dx^1 dx^2 dx^3 dx^4 = 0. \quad (96)$$

§ 6. EQUATIONS OF MOTION FOR AN ISOLATED BODY

We return to the case of an isolated body, as considered in IV-§ 7. Its history is a world-tube with

$$T^{ij} N_j = 0 \quad (97)$$

on the wall Σ of the tube, and

$$T^{ij}_{|j} = 0 \quad (98)$$

inside the tube. In view of the field equations, it is obviously a matter of indifference whether we work with the Einstein tensor G_{ij} (geometry) or with the energy tensor T_{ij} (physics). We shall use T_{ij} .

Proceeding in the Euclidean manner of the preceding section, we

write out explicitly the tensorial equation (98):

$$T^{ij}_{,j} + \Gamma^j_{aj} T^{ia} + \Gamma^i_{aj} T^{aj} = 0. \quad (99)$$

Since

$$\Gamma^j_{aj} = \frac{1}{\sqrt{-g}} (\sqrt{-g})_{,a}, \quad (100)$$

(99) may be written

$$\mathcal{T}^{ij}_{,j} = \Gamma^i, \quad (101)$$

where

$$\mathcal{T}^{ij} = \sqrt{-g} T^{ij}, \quad (102)$$

and

$$\Gamma^i = -\Gamma^i_{ab} \mathcal{T}^{ab}. \quad (103)$$

Although we are deliberately abandoning tensorial ideas, we remark in passing that \mathcal{T}^{ij} is a tensor density or relative tensor of weight unity (cf. SYNGE and SCHILD [1956, pp. 198, 241]).

We now slice the world-tube across by two planes, $x^4 = a$ and $x^4 = b$ (Fig. 7), and denote by V_4 the domain of space-time enclosed by them and Σ . By Green's theorem in the form (95), we have

$$\int_{V_4} \mathcal{T}^{ij}_{,j} d^4x = \oint_{V_3} \mathcal{T}^{ij} \nu(f_{,j}/f_{,4}) d^3x, \quad (104)$$

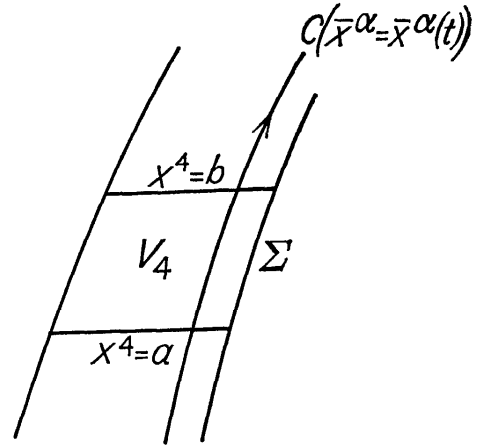


Fig. 7 – World-tube sliced in Euclidean manner

where we have written for brevity

$$d^4x = dx^1 dx^2 dx^3 dx^4, \quad d^3x = dx^1 dx^2 dx^3; \quad (105)$$

$f(x) = 0$ is the equation of the closed V_3 , made up of Σ and the two plane sections. Now, in view of (97), we have

$$\begin{aligned} \mathcal{T}^{ij} f_{,j} &= 0 \text{ on } \Sigma, \\ f_{,j}/f_{,4} &= \delta^j_4 \text{ on } x^4 = a \text{ and } x^4 = b, \\ \nu &= 1 \text{ on } x^4 = b, \quad \nu = -1 \text{ on } x^4 = a. \end{aligned} \quad (106)$$

Therefore (104) gives

$$\int_{V_4} \mathcal{T}^{ij}_{,j} d^4x = \int_{x^4=b} \mathcal{T}^{i4} d^3x - \int_{x^4=a} \mathcal{T}^{i4} d^3x, \quad (107)$$

or, by (101),

$$\int_{x^4=b} \mathcal{T}^{i4} d^3x - \int_{x^4=a} \mathcal{T}^{i4} d^3x = \int_{V_4} \Gamma^i d^4x. \quad (108)$$

Dividing by $(b - a)$ and proceeding to the limit $b \rightarrow a$, we get

$$\frac{d}{dx^4} \int \mathcal{T}^{i4} d^3x = \int \Gamma^i d^3x, \quad (109)$$

the integrals being taken on any section $x^4 = \text{const.}$

In like manner we shall derive another formula. We have

$$(x^k \mathcal{T}^{ij})_{,j} = \mathcal{T}^{ik} + x^k \mathcal{T}^{ij}_{,j} = \mathcal{T}^{ik} + x^k \Gamma^i, \quad (110)$$

and hence

$$\int_{x^4=b} x^k \mathcal{T}^{i4} d^3x - \int_{x^4=a} x^k \mathcal{T}^{i4} d^3x = \int_{V_4} (\mathcal{T}^{ik} + x^k \Gamma^i) d^4x. \quad (111)$$

Giving k a value in the range 1, 2, 3 (indicated by Greek suffixes) and proceeding to a limit as above, we get

$$\frac{d}{dx^4} \int x^\alpha \mathcal{T}^{i4} d^3x = \int (\mathcal{T}^{i\alpha} + x^\alpha \Gamma^i) d^3x. \quad (112)$$

With $i = 4$ this gives

$$\frac{d}{dx^4} \int x^\alpha \mathcal{T}^{44} d^3x = \int (\mathcal{T}^{4\alpha} + x^\alpha \Gamma^4) d^3x. \quad (113)$$

On the other hand, if we put β for i in (112), interchange α and β , and subtract, remembering the symmetry of \mathcal{T}^{ij} , we get

$$\frac{d}{dx^4} \int (x^\alpha \mathcal{T}^{\beta 4} - x^\beta \mathcal{T}^{\alpha 4}) d^3x = \int (x^\alpha \Gamma^\beta - x^\beta \Gamma^\alpha) d^3x. \quad (114)$$

In order to write the equations so far obtained more compactly we shall introduce new notation, and to enhance the physical interest we shall insert some names. Thus we define ¹

$$\text{4-momentum of body} = M^i = \int \mathcal{T}^{i4} d^3x,$$

$$\text{angular momentum of body} = H^{\alpha\beta} = \int (x^\alpha \mathcal{T}^{\beta 4} - x^\beta \mathcal{T}^{\alpha 4}) d^3x, \quad (115)$$

$$\text{mass-centre of body} = \bar{x}^\alpha, \quad M^4 \bar{x}^\alpha = \int x^\alpha \mathcal{T}^{44} d^3x.$$

¹ We are following in the main, but not in all subsequent details, the method of LANCZOS [1941b]. The verbal definitions of (115) are for present purposes only; it is contrary to the spirit of this book to attach physical names to quantities which are not invariantly defined.

We note that

$$\int (x^\alpha - \bar{x}^\alpha) \mathcal{T}^{44} d^3x = 0. \quad (116)$$

This expression transforms like a 3-vector under rotation of axes, and if we make the natural assumption [cf. IV-(146a)]

$$\mathcal{T}^{44} > 0, \quad (117)$$

it follows that every plane through the mass-centre cuts the body; hence, if the body is convex, its mass-centre lies in it.

If we now write $x^4 = t$, the equations (109) and (114) give

$$\begin{aligned} \frac{dM^i}{dt} &= \int \Gamma^i d^3x, \\ \frac{dH^{\alpha\beta}}{dt} &= \int (x^\alpha \Gamma^\beta - x^\beta \Gamma^\alpha) d^3x. \end{aligned} \quad (118)$$

Also, by (113),

$$\frac{d}{dt} (M^4 \bar{x}^\alpha) = M^\alpha + \int x^\alpha \Gamma^4 d^3x, \quad (119)$$

and this gives the following expression for the 3-velocity of the mass-centre:

$$\frac{d\bar{x}^\alpha}{dt} = \frac{M^\alpha}{M^4} + \frac{1}{M^4} \int (x^\alpha - \bar{x}^\alpha) \Gamma^4 d^3x. \quad (120)$$

The remarkable feature of the above work is that, by breaking away from the restriction of tensorial invariance, we are led to some very simple equations which are physically suggestive. For in (118) the rates of change of 4-momentum and angular momentum are expressed in terms of quantities which may be regarded as the gravitational 4-force and torque acting on the body, in the sense that, if there is no gravitational field and coordinates are chosen to make Γ_{jk}^i vanish, then this 4-force and torque vanish also. Further, if we neglect the last term in (120), this equation tells us that the 4-momentum M^i points along the 4-velocity \bar{V}^i of the mass-centre.

But where do we stand in regard to invariance? We started with geometric objects — a world-tube, a metric tensor, and a symmetric tensor T^{ij} satisfying (97) and (98). But when we sliced the world-tube

across, we made all subsequent results dependent on the mode of slicing, and, further, we tied the coordinates into the work in a non-invariant manner. Since, subject only to certain general restrictions, all systems of coordinates are available, we must recognize that we have before us, not *one* world-line C for the mass-centre, but a multiplicity of such world-lines. Likewise we have not *one* 4-momentum M^i and *one* angular momentum $H^{\alpha\beta}$, but a multiplicity of these. There opens up, then, an interesting prospect — to restore uniqueness and invariance to the results by some statistical process in which all possible coordinate-choices are taken into consideration. But that is an ambitious programme, not to be attempted here.

Instead we shall play with the exact equations as given above, seeking approximations by which we can force out a ‘proof’ of the geodesic hypothesis by showing that the mass-centre of a very small body pursues a geodesic.

In making approximations, it is well to examine the dimensionalities of the quantities involved ¹. Mass, length and time all have the same dimensions, say $[t]$. Let us use coordinates with dimensions $[t]$. Then

$$\begin{aligned} [g_{ij}] &= [t^0], & [F^i_{jk}] &= [t^{-1}], & [T^{ij}] &= [\mathcal{T}^{ij}] = [t^{-2}], \\ [F^i] &= [t^{-3}], & [\int F^i d^3x] &= [t^0], & [M^i] &= [t], \\ [\int (x^\alpha - \bar{x}^\alpha) F^4 d^3x] &= [t]. \end{aligned} \quad (121)$$

The last term in (120) is dimensionless, and tends to zero as the size of the body tends to zero, other things being equal. Further, for a weak field, we may suppose F^i_{jk} to be small, in a loose manner of speaking. It is therefore reasonable, in the case of a very small body, to neglect this term, so that (120) reads

$$\frac{d\bar{x}^\alpha}{dt} = \frac{M^\alpha}{M^4}. \quad (122)$$

The next step is a little more dubious. We want to treat F^i_{jk} as constant over the section $t = \text{const}$. To justify this, we say that if \mathcal{T}^{ij} is small (relative to what?) the body has very little effect on the field, thought of as primarily due to other unspecified bodies, and so F^i_{jk} will change little across a small section. Hence we write (note that this is a

¹ Cf. remarks about smallness in II-§ 3.

dimensionless quantity)

$$\int I^i d^3x = - \int I_{mn}^i \mathcal{T}^{mn} d^3x = - \bar{I}_{mn}^i \phi^{mn}, \quad (123)$$

where

$$\phi^{mn} = \int \mathcal{T}^{mn} d^3x, \quad \phi^{i4} = M^i, \quad (124)$$

with \bar{I}_{mn}^i evaluated at the mass-centre. Thus (118) gives

$$\frac{dM^i}{dt} = - \bar{I}_{mn}^i \phi^{mn}, \quad (125)$$

and, by differentiation of (122),

$$\begin{aligned} \frac{d^2 \bar{x}^\alpha}{dt^2} &= (M^4)^{-2} (\bar{I}_{mn}^4 M^\alpha - \bar{I}_{mn}^\alpha M^4) \phi^{mn} \\ &= (M^4)^{-1} \left(\bar{I}_{mn}^4 \frac{d\bar{x}^\alpha}{dt} - \bar{I}_{mn}^\alpha \right) \phi^{mn}, \end{aligned} \quad (126)$$

or

$$M^4 \frac{d^2 \bar{x}^\alpha}{dt^2} + \bar{I}_{mn}^\alpha \phi^{mn} = \bar{I}_{mn}^4 \phi^{mn} \frac{d\bar{x}^\alpha}{dt}. \quad (127)$$

These are in a sense *equations of motion of the mass-centre*.

In terms of density, 4-velocity and stress, we have

$$\mathcal{T}^{ij} = \sqrt{-g} (\mu V^i V^j - S^{ij}). \quad (128)$$

Let us suppose that the following equations are so nearly satisfied that it is permissible to use them in (127):

$$\int \mu V^i V^j \sqrt{-g} d^3x = \bar{V}^i \bar{V}^j \int \mu \sqrt{-g} d^3x, \quad (129)$$

$$\int S^{ij} \sqrt{-g} d^3x = 0; \quad (130)$$

here \bar{V}^i denotes the 4-velocity of the mass-centre, so that

$$\frac{d\bar{x}^i}{dt} = \frac{\bar{V}^i}{\bar{V}^4}. \quad (131)$$

Then by (124)

$$\phi^{mn} = \bar{V}^m \bar{V}^n \int \mu \sqrt{-g} d^3x, \quad M^4 = \phi^{44}. \quad (132)$$

If we define $\bar{x}^4 = t$, and write 4 instead of α in (127), we get an identity.

It is permissible then to change α to i in (127); with (131), this gives

$$(\bar{V}^4)^2 \frac{d}{dt} \frac{\bar{V}^i}{\bar{V}^4} + \bar{\Gamma}_{mn}^i \bar{V}^m \bar{V}^n = \bar{\Gamma}_{mn}^4 \bar{V}^m \bar{V}^n \frac{\bar{V}^i}{\bar{V}^4}. \quad (133)$$

This means that, in terms of the absolute derivative,

$$\bar{V}^4 \frac{\delta \bar{V}^i}{\delta t} = \bar{V}^i \frac{\delta \bar{V}^4}{\delta t}. \quad (134)$$

Multiplying by \bar{V}_i , we get $\delta \bar{V}^4 / \delta t = 0$, and hence we have

$$\frac{\delta \bar{V}^i}{\delta t} = 0, \quad (135)$$

which tells us that, *under the assumptions made, the world-line of the mass-centre is a geodesic.*

It is hard to see whether anything significant has been established here. We know that if we put $S_{ij} = 0$ we have incoherent matter, and for incoherent matter the world-lines are geodesics (cf. IV-§ 4). The condition (130) is somewhat weaker, but this assumption and (129) are too empirical for us to regard this argument as a *proof* that the world-line of a small isolated body is a geodesic ¹.

§ 7. THE PSEUDO-TENSOR

In space-time, viewed from the Euclidean standpoint as in § 5, let there be a symmetric array of quantities W^{ik} ($= W^{ki}$) which satisfy the partial differential equations

$$W^{ik}_{,k} = 0. \quad (136)$$

With regard to coordinate transformations, we can, if we like, disregard them entirely, accepting (136) as true for some given coordinates x^i and working entirely in those coordinates. But in the theory of the pseudo-tensor ² to be developed below, the transformation law for W^{ik} is such that the *equation* (136) holds for all coordinates, although W^{ik} is not a tensor, nor is $W^{ik}_{,k}$ a vector in the tensorial sense.

¹ I find it difficult to enter into the spirit of the work of INFELD and SCHILD [1949], PAPAPETROU [1951b], CORINALDESI and PAPAPETROU [1951].

² The pseudo-tensor was introduced by EINSTEIN [1916a]; cf. BERGMANN [1942], MØLLER [1952]. We follow here a different approach presented by LANDAU and LIFSHITZ [1951, p. 316]. As its name implies, the pseudo-tensor is *not* a tensor and the conservation laws based on it are not tensorial.

Integrated over the infinite 3-space $x^4 = \text{const.}$, (136) gives

$$\int W^{ik},_k d^3x = 0, \quad d^3x = dx^1 dx^2 dx^3. \quad (137)$$

Supposing W^{ik} to vanish sufficiently rapidly at spatial infinity, the contributions from $k = 1, 2, 3$ vanish, and (137) gives

$$\frac{d}{dx^4} \int W^{i4} d^3x = 0, \quad (138)$$

so that, considering all the slices $x^4 = \text{const.}$, we have

$$\int W^{i4} d^3x = \kappa M^i, \quad (139)$$

where M^i are four constants, independent of x^4 . The constant $\kappa (= 8\pi)$ is inserted merely to simplify later formulae.

So far we have not used the *symmetry* of W^{ik} . By virtue of it, we have

$$(x^i W^{jk} - x^j W^{ik}),_k = 0, \quad (140)$$

and an argument similar to the above leads to

$$\int (x^i W^{j4} - x^j W^{i4}) d^3x = \kappa H^{ij}, \quad (141)$$

where $H^{ij} (= -H^{ji})$ are six constants, independent of x^4 .

We have now to choose W^{ik} so that (139) and (141) may qualify for the title 'conservation laws' — (139) for 4-momentum and (141) for angular momentum. To make this choice, we use the following mathematical identity ¹:

$$gG^{ik} = -\frac{1}{2}U^{ijkm}_{,jm} + V^{ik}. \quad (142)$$

Here G^{ik} is the contravariant form of the Einstein tensor and

$$U^{ijkm} = g(g^{ik}g^{jm} - g^{im}g^{jk}). \quad (143)$$

As for V^{ik} , it is a complicated expression which however depends only on the metric tensor and its *first* derivatives. We shall calculate it later, but first let us see how (142) is to be used.

We observe that U^{ijkm} has the same symmetry properties as the Riemann tensor R^{ijkm} ; from this it is easy to see that V^{ik} is symmetric, and, further, that

$$U^{ijkm}_{,jmk} = 0. \quad (144)$$

This is an essential point. By virtue of it, W^{ik} defined by

$$W^{ik} = -gG^{ik} + V^{ik} = \frac{1}{2}U^{ijkm}_{,jm} \quad (145)$$

¹ This may be regarded as the definition of V^{ik} .

is not only symmetric but satisfies (136), and hence (139) and (141) also (provided the required conditions at infinity are satisfied, as we shall suppose they are). Noting the field equations

$$G^{ik} = -\kappa T^{ik}, \quad \kappa = 8\pi, \quad (146)$$

we define the *pseudo-tensor of energy* t^{ik} by

$$t^{ik} = \kappa^{-1} g^{-1} V^{ik}, \quad (147)$$

so that (145) reads

$$W^{ik} = \kappa g(T^{ik} + t^{ik}). \quad (148)$$

Now (139) and (141) give

$$\int g(T^{i4} + t^{i4}) d^3x = M^i, \quad (149)$$

$$\int g[x^i(T^{j4} + t^{j4}) - x^j(T^{i4} + t^{i4})] d^3x = H^{ij}, \quad (150)$$

M^i and H^{ij} being constants independent of x^4 . The form of these equations suggests that they should be regarded as equations of conservation of 4-momentum (M^i) and angular momentum (H^{ij}). Under certain conditions at infinity which we have not troubled to specify in detail, they are mathematically correct. But it is a little difficult to accept the physical interpretation, since, although true for any chosen coordinate system, the constants M^i and H^{ij} change with change of coordinates in no simple way. The sections $x^4 = \text{const.}$ used in the integrations depend of course on the choice of coordinates.

It remains to calculate V^{ik} in (142). For this we use the following identities [cf. I-(88) and I-(105)]:

$$\begin{aligned} \Gamma_{ab,c}^i - \Gamma_{ac,b}^i &= R_{.acb}^i - \Gamma_{ab}^p \Gamma_{pc}^i + \Gamma_{ac}^p \Gamma_{pb}^i, \\ \Gamma_{pb,a}^p - \Gamma_{ab,p}^p &= R_{ab} - \Gamma_{aq}^p \Gamma_{bp}^q + \Gamma_{ab}^p \Gamma_{pq}^q. \end{aligned} \quad (151)$$

From (143) we have

$$(g^{-1} U^{ijkm})_{|p} = 0, \quad (152)$$

and hence, using I-(8),

$$\begin{aligned} U^{ijkm}_{,p} &= 2\Gamma_{ap}^a U^{ijkm} - \Gamma_{ap}^i U^{ajkm} - \Gamma_{ap}^j U^{iakm} \\ &\quad - \Gamma_{ap}^k U^{ijam} - \Gamma_{ap}^m U^{ijka}. \end{aligned} \quad (153)$$

Putting $p = j$ and using the symmetry of U^{ijkm} , we get

$$U^{ijkm}_{,j} = \Gamma_{aj}^a U^{ijkm} - \Gamma_{aj}^k U^{ijam} - \Gamma_{aj}^m U^{ijka}, \quad (154)$$

and differentiation of this gives

$$U^{ijkm}_{,jm} = A^{ik} + B^{ik}, \quad (155)$$

where

$$\begin{aligned} A^{ik} &= \Gamma^a_{aj,m} U^{ijkm} - \Gamma^k_{aj,m} U^{ijam} - \Gamma^m_{aj,m} U^{ijka}, \\ B^{ik} &= \Gamma^a_{aj} U^{ijkm}_{,m} - \Gamma^k_{aj} U^{ijam}_{,m} - \Gamma^m_{aj} U^{ijka}_{,m}. \end{aligned} \quad (156)$$

Note that second derivatives of g_{ij} are contained in A^{ik} but not in B^{ik} . We may write

$$A^{ik} = U^{ijkm}(\Gamma^p_{pj,m} - \Gamma^p_{mj,p}) + \frac{1}{2}U^{ijam}(\Gamma^k_{jm,a} - \Gamma^k_{ja,m}), \quad (157)$$

and so, by (151),

$$\begin{aligned} A^{ik} &= U^{ijkm}R_{jm} + \frac{1}{2}U^{ijam}R^k_{jam} \\ &+ U^{ijkm}(\Gamma^p_{mj}\Gamma^q_{pq} - \Gamma^p_{mq}\Gamma^q_{jp}) + \frac{1}{2}U^{ijam}(\Gamma^p_{ja}\Gamma^k_{pm} - \Gamma^p_{jm}\Gamma^k_{pa}). \end{aligned} \quad (158)$$

But

$$\begin{aligned} U^{ijkm}R_{jm} &= g(g^{ik}g^{jm} - g^{im}g^{jk})R_{jm} = g(g^{ik}R - R^{ik}), \\ U^{ijam}R^k_{jam} &= g(g^{ia}g^{jm} - g^{im}g^{ja})R^k_{jam} = -2gR^{ik}, \end{aligned} \quad (159)$$

and so (155) may be written

$$U^{ijkm}_{,jm} = -2gG^{ik} + 2V^{ik}, \quad (160)$$

where

$$2V^{ik} = U^{ijkm}(\Gamma^p_{mj}\Gamma^q_{pq} - \Gamma^p_{mq}\Gamma^q_{jp}) + U^{ijam}\Gamma^p_{ja}\Gamma^k_{pm} + B^{ik}. \quad (161)$$

It is clear that V^{ik} does not involve the second derivatives of g_{ij} . To complete the calculation, we evaluate the derivatives in B^{ik} by means of (153) and (154), obtaining

$$\begin{aligned} 2V^{ik} &= U^{iakb}D_{ab} + U^{iabc}E^k_{abc} + U^{kabc}E^i_{abc} + U^{abcd}\Gamma^i_{ad}\Gamma^k_{bc}, \\ D_{ab} &= \Gamma^p_{pa}\Gamma^q_{qb} + \Gamma^p_{qa}\Gamma^q_{pb} - 2\Gamma^p_{ab}\Gamma^q_{pq}, \\ E^k_{abc} &= -\Gamma^k_{ab}\Gamma^p_{pc} - \Gamma^p_{ab}\Gamma^k_{pc}. \end{aligned} \quad (162)$$

CHAPTER VII

FIELDS WITH SPHERICAL SYMMETRY

§ 1. SPACE-TIME OF CONSTANT CURVATURE (DE SITTER UNIVERSE)

Of all Riemann space-times, the simplest is the flat space-time of Minkowski. It corresponds physically to the complete absence of gravitation, and is the domain in which the special theory of relativity is set. In the general theory of relativity we have little interest in flat space-time except in connection with conditions at infinity (we may suppose that a gravitational field tends to zero at infinity) and in connection with plane gravitational waves which will be discussed in Chapter IX (space-time is flat outside the wave).

Next in order of simplicity comes space-time of constant curvature. If K denotes the constant curvature, we have by I-(101)

$$\begin{aligned} R_{ijkm} &= K(g_{ik}g_{jm} - g_{im}g_{jk}), \\ R_{ij} &= -3Kg_{ij}, \\ R &= -12K, \\ G_{ij} &= 3Kg_{ij}. \end{aligned} \tag{1}$$

By the field equations IV-(108) with the cosmological constant included, we have

$$G_{ij} - \Lambda g_{ij} = -\kappa T_{ij}, \quad \kappa = 8\pi, \tag{2}$$

and so the energy tensor in space-time of constant curvature K is

$$T_{ij} = \kappa^{-1}(\Lambda - 3K)g_{ij}. \tag{3}$$

This tensor has all four eigenvalues equal and completely indeterminate eigenvectors. It corresponds to no reasonable type of matter, and we escape from this awkward situation only by assuming that the cosmological constant Λ and the constant curvature K are related by the

equation ¹

$$\Lambda = 3K. \quad (4)$$

This makes $T_{ij} = 0$ and we have the empty universe of DE SITTER [1917a], satisfying the field equations

$$G_{ij} - \Lambda g_{ij} = 0. \quad (5)$$

It is hard to decide on the degree of seriousness with which we should contemplate, as physicists, the empty universe of de Sitter. Since we have thrown out matter, we are back in what is essentially a universe without gravitation, with Minkowskian space-time replaced by space-time of constant curvature. The success of the special theory of relativity (flat space-time) in dealing with those phenomena which do not involve gravitation suggests that, if we are to work instead with a de Sitter universe, the curvature must be very small indeed in comparison with significant physical quantities of like dimensions (K has the dimensions of sec^{-2}). Without good reason one does not feel inclined to complicate the simplicity of Minkowskian space-time by introducing curvature.

Nevertheless the de Sitter universe is interesting in itself. It opens up new vistas, introducing us to the idea that space (a slice of space-time) may be finite, and this seems to satisfy some mental need in us, for infinity is one of those things which we find difficulty in comprehending.

To explore the de Sitter universe, we choose some event P' and draw through P' all geodesics, timelike, spacelike and null (Fig. 1). On each geodesic we choose a special parameter u with $u = 0$ at P' so that we

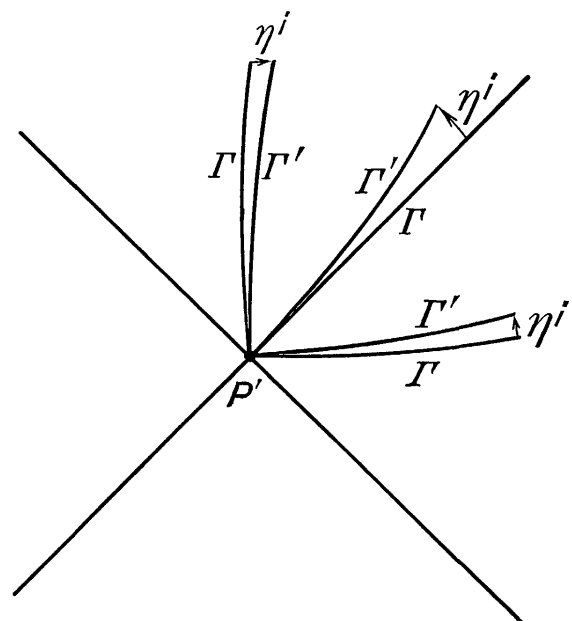


Fig. 1 – Exploring the de Sitter universe

¹ It is customary to regard the cosmological constant as positive, and hence K as positive also. However, the idea of a space-time of constant curvature of either sign is stimulating and worth exploring, and for simplicity we shall speak of a de Sitter universe no matter which sign K has. We shall see later that a negative K has a consequence so strange that we can hardly accept a universe of that type as a model of physical reality.

have

$$\frac{\delta U^i}{\delta u} = 0, \quad U^i = \frac{dx^i}{du}, \quad (6)$$

and we set into correspondence the events on two neighbouring geodesics Γ , Γ' by choosing them at equal values of u . Then the deviation vector η^i satisfies the equation 1-(131):

$$\frac{\delta^2 \eta^i}{\delta u^2} + R^i_{jkm} U^j \eta^k U^m = 0, \quad (7)$$

with

$$\eta^i = 0 \text{ for } u = 0. \quad (8)$$

Substituting from (1), we get

$$\frac{\delta^2 \eta^i}{\delta u^2} + \eta^i K U_j U^j - U^i K U_j \eta^j = 0. \quad (9)$$

To study the deviation of spacelike or timelike geodesics, we choose $u = s$, and then by 1-(133) and (8) we have

$$U_j U^j = \varepsilon, \quad U_j \eta^j = 0, \quad (10)$$

where ε is the indicator of Γ ; (9) reduces to

$$\frac{\delta^2 \eta^i}{\delta s^2} + \varepsilon K \eta^i = 0. \quad (11)$$

Introducing any vector λ^i undergoing parallel transport on Γ , we get

$$\frac{d^2}{ds^2} (\eta^i \lambda_i) + \varepsilon K \eta^i \lambda_i = 0, \quad (12)$$

and the solution of this elementary equation can be written down at once. For example, if $K > 0$ and $\varepsilon = 1$, we have

$$\eta^i \lambda_i = A \sin(sK^{\frac{1}{2}}), \quad (13)$$

where A is a constant. If we define ζ^i by

$$\zeta^i = \frac{\delta \eta^i}{\delta u}, \quad (14)$$

and use primes to refer to P' , then, in terms of the parallel propagator,

(13) may be written

$$\eta^i = K^{-\frac{1}{2}} g^{ij'} \zeta_{j'} \sin(sK^{\frac{1}{2}}). \quad (15)$$

Thus, taking all cases into consideration, the deviations of spacelike and timelike geodesics are as follows:

Space-time of positive curvature ($K > 0$):

$$\text{Spacelike geodesics } (\varepsilon = 1): \quad \eta^i = K^{-\frac{1}{2}} g^{ij'} \zeta_{j'} \sin(sK^{\frac{1}{2}}). \quad (16)$$

$$\text{Timelike geodesics } (\varepsilon = -1): \quad \eta^i = K^{-\frac{1}{2}} g^{ij'} \zeta_{j'} \sinh(sK^{\frac{1}{2}}). \quad (17)$$

Space-time of negative curvature ($K < 0$):

$$\text{Spacelike geodesics } (\varepsilon = 1): \quad \eta^i = (-K)^{-\frac{1}{2}} g^{ij'} \zeta_{j'} \sinh[s(-K)^{\frac{1}{2}}]. \quad (18)$$

$$\text{Timelike geodesics } (\varepsilon = -1): \quad \eta^i = (-K)^{-\frac{1}{2}} g^{ij'} \zeta_{j'} \sin[s(-K)^{\frac{1}{2}}]. \quad (19)$$

It remains to consider deviation when one (or both) of the curves Γ, Γ' is (or are) null. Taking first the case where both are null, we have, instead of (10),

$$U_j U^j = 0, \quad U_j \eta^j = 0, \quad (20)$$

so that (9) becomes simply

$$\frac{\delta^2 \eta^i}{\delta u^2} = 0, \quad (21)$$

and the deviation is given by

$$\eta^i = u g^{ij'} \zeta_{j'}. \quad (22)$$

Now suppose Γ to be null but Γ' spacelike or timelike. We have the first of (20) but not the second. Multiplying (9) by U_i , we get

$$-\frac{d^2}{du^2} (\eta^i U_i) = 0, \quad (23)$$

and so

$$\eta^i U_i = au, \quad a = \zeta^{i'} U_{i'}. \quad (24)$$

Thus (9) reads

$$\frac{\delta^2 \eta^i}{\delta u^2} - u U^i K a = 0, \quad (25)$$

and so, with a parallel vector λ^i ,

$$\frac{d^2}{du^2} (\eta^i \lambda_i) = u K a U^i \lambda_i. \quad (26)$$

The right hand side is constant, and we get for the deviation

$$\eta^i = u g^{ij} \zeta_{j'} + \frac{1}{6} u^3 K U^i \zeta^{k'} U_{k'}. \quad (27)$$

The preceding calculations are based on one assumption only — space-time is of constant curvature K . This assumption does not determine space-time completely, since it does not include a topological specification. To illustrate by a simple analogy, the fact that a 2-space with positive-definite metric is flat does not imply that it is a plane: a cylinder is flat, and there exists a flat 2-space (the product of two circles) which has the topology of the torus. Nevertheless we can extract some interesting information from the formulae (16)–(19).

Suppose $K > 0$. Then (16) tells us that two adjacent spacelike geodesics drawn from any event P' meet again ($\eta^i = 0$) at an event P where

$$s = \pi K^{-\frac{1}{2}}. \quad (28)$$

Indeed, passing from neighbour to neighbour, we see that *all* the spacelike geodesics drawn from P' meet at the single event P (Fig. 2), the lengths of them all being the same, as in (28). If, having arrived at

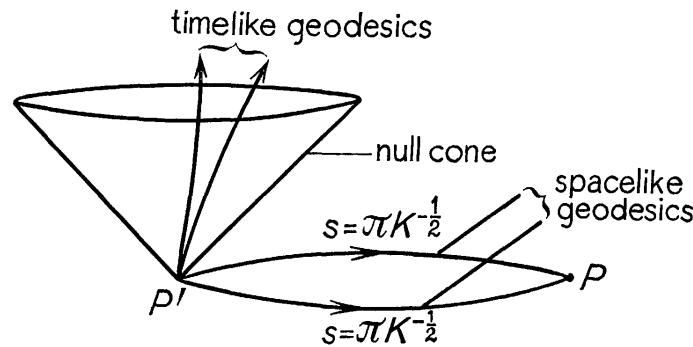


Fig. 2 – Geodesics in de Sitter universe with $K > 0$

P by one of these geodesics, we carry on through P , we shall get back to P' after a further distance (28). Thus all spacelike geodesics are closed curves. However, two cases arise. In the first case P is distinct from P' (think of the poles of the earth), and the length of a closed geodesic is $2\pi K^{-\frac{1}{2}}$; this is called the *antipodal* or *spherical* case. In the second case, P is P' , so that $P'P$ is itself a closed geodesic of length $\pi K^{-\frac{1}{2}}$; this is called the *polar* or *elliptic* case. As for two adjacent timelike geodesics, (17) tells us that they open out exponentially and never intersect again, unless possibly by virtue of some topological condition which might be imposed.

If $K < 0$, the roles of the spacelike and timelike geodesics are interchanged (Fig. 3). The spacelike geodesics open out exponentially, whereas the timelike geodesics from an event P' meet again at an event P after a time

$$s = \pi(-K)^{-\frac{1}{2}}; \quad (29)$$

these results follow from (18) and (19). This meeting of the timelike geodesics is a strange matter, and it is best to say no more about it

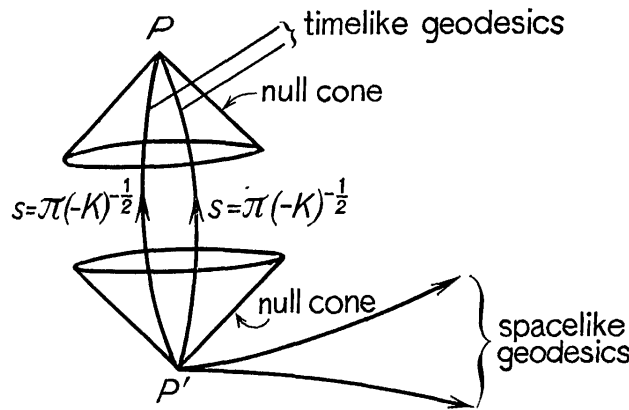


Fig. 3 – Geodesics in de Sitter universe with $K < 0$

just at present because the deviation method is really not adequate to deal with it properly.

There are various ways of discussing the de Sitter universe ¹, each with its own particular interest, but, in order to clarify questions of topology, it is best to construct a de Sitter universe as a 4-space V_4 embedded in a flat 5-space V_5 which has Euclidean topology. Let capital suffixes take the values 1, 2, 3, 4, 5. Consider a flat V_5 with coordinates x^A , each of which runs from $-\infty$ to $+\infty$ and a metric form

$$\Psi = \eta_{AB} dx^A dx^B, \quad (30)$$

where η_{AB} is a diagonal 5×5 matrix with elements ± 1 — we do not specify them further yet. When we say that V_5 has Euclidean topology we mean simply that there is a 1 : 1 correspondence between the points of V_5 and the pentads x^A (in fact, a point is a pentad). We define V_4 by the equation

$$\eta_{AB} x^A x^B = C, \quad (31)$$

¹ Cf. SCHRÖDINGER [1956].

where C is a constant. This may also be written

$$(x^5)^2 = \eta_{55}(C - \eta_{ij}x^i x^j), \quad (32)$$

and so, for any displacement in V_4 , we have

$$x^5 dx^5 = -\eta_{55} \eta_{ij} x^i dx^j. \quad (33)$$

The next step is to calculate the curvature of V_4 . This is a local matter, and we shall use x^i as coordinates without being troubled by the fact that (32) gives two values to x^5 . The metric induced in V_4 by the metric (30) is

$$\Phi = \eta_{ij} dx^i dx^j + (C - S)^{-1} (\eta_{ij} x^i dx^j)^2 = g_{ij} dx^i dx^j, \quad (34)$$

where

$$\begin{aligned} g_{ij} &= \eta_{ij} + y_i y_j (C - S)^{-1}, \\ y_i &= \eta_{ik} x^k, \quad S = \eta_{ij} x^i x^j = y_i x^i = \eta_{ij} y_i y_j. \end{aligned} \quad (35)$$

In g_{ij} we have the metric tensor of V_4 , and it is easy to prove by matrices (or to verify directly) that the conjugate tensor is

$$g^{ij} = \eta_{ij} - x^i x^j C^{-1}. \quad (36)$$

The 4-space V_4 may, or may not, contain a point of V_5 for which $x^i = 0$. If it does, we see from (35) that we should choose

$$\eta_{ij} = \text{diag}(1, 1, 1, -1) \quad (37)$$

in order that the metric of V_4 may have the correct signature for space-time.

It is now easy to calculate the Christoffel symbols and hence, by I-(88), the Riemann tensor for V_4 . We obtain

$$\begin{aligned} \Gamma_{ij}^a &= C^{-1} x^a [\eta_{ij} + (C - S)^{-1} y_i y_j], \\ R_{ijklm} &= C^{-1} (g_{ik} g_{jm} - g_{im} g_{jk}). \end{aligned} \quad (38)$$

Hence, by I-(101), V_4 is a de Sitter universe in the sense that it is 4-space of constant curvature

$$K = C^{-1}. \quad (39)$$

Having settled, as above, the question of topology, we can explore

this embedded de Sitter universe without confusion ¹. We shall here accept, without giving a formal proof, that the geodesics of V_4 are (like the great circles on an ordinary sphere) the intersections of V_4 with 2-flats passing through the origin of V_5 . This makes their discussion very simple if we use a vector notation, writing $P^A = \mathbf{P}$, $x^A = \mathbf{x}$, and using the scalar product

$$\mathbf{P} \cdot \mathbf{Q} = \eta_{AB} P^A Q^B. \quad (40)$$

By (31) and (39), the equation of V_4 is

$$\mathbf{x} \cdot \mathbf{x} = K^{-1}. \quad (41)$$

Then, for an infinitesimal displacement in V_4 we have

$$\mathbf{x} \cdot d\mathbf{x} = 0. \quad (42)$$

Let $\mathbf{x} = \mathbf{P}$ be any point on V_4 and Γ a geodesic of V_4 drawn in the direction of a vector \mathbf{Q} . We have then

$$\mathbf{P} \cdot \mathbf{P} = K^{-1}, \quad \mathbf{P} \cdot \mathbf{Q} = 0. \quad (43)$$

The 2-flat through \mathbf{P} and \mathbf{Q} has the parametric equation

$$\mathbf{x} = p\mathbf{P} + q\mathbf{Q}, \quad (44)$$

with p and q running through all values. Now Γ is the intersection of (41) and (44). In fact, (44) is the parametric equation of Γ with p and q satisfying

$$(p\mathbf{P} + q\mathbf{Q}) \cdot (p\mathbf{P} + q\mathbf{Q}) = K^{-1}, \quad (45)$$

with $p = 1, q = 0$ at \mathbf{P} . We have

$$p^2 + K\mathbf{Q} \cdot \mathbf{Q}q^2 = 1. \quad (46)$$

Suppose that Γ is a null geodesic (null, that is, in V_5 and hence in V_4 too in terms of the induced metric). Then

$$\mathbf{Q} \cdot \mathbf{Q} = 0, \quad p = 1, \quad (47)$$

and the parametric equation of Γ reads

$$\mathbf{x} = \mathbf{P} + q\mathbf{Q} \quad (-\infty < q < \infty). \quad (48)$$

¹ In order not to burden the argument, we restrict the discussion to antipodal cases; to get a polar case, we would identify diametrically opposed points on V_4 , i.e. points lying on a straight line of V_5 passing through the origin of V_5 .

The null geodesics of V_4 are in fact straight lines in V_5 (analogous to the generators on an ordinary hyperboloid of one sheet).

Now suppose that Γ is not null. We can normalize \mathbf{Q} so that

$$K\mathbf{Q} \cdot \mathbf{Q} = \omega = \pm 1, \quad (49)$$

and (46) reads

$$p^2 + \omega q^2 = 1. \quad (50)$$

The sign of ω is most important. If $\omega = 1$, then we can define u by

$$\cos u = p, \quad \sin u = q, \quad (51)$$

and (44) gives as parametric equation of Γ

$$\mathbf{x} = \mathbf{P} \cos u + \mathbf{Q} \sin u. \quad (52)$$

This is in fact a circle in V_5 . When $u = \pi$, we get $\mathbf{x} = -\mathbf{P}$; when $u = 2\pi$, we get $\mathbf{x} = \mathbf{P}$. These geodesics are closed curves. All geodesics of this type which start from \mathbf{P} meet at the antipode $-\mathbf{P}$ and again at \mathbf{P} itself. On the other hand, if $\omega = -1$, we can define u by

$$\cosh u = p, \quad \sinh u = q, \quad (53)$$

and the equation of Γ reads

$$\mathbf{x} = \mathbf{P} \cosh u + \mathbf{Q} \sinh u. \quad (54)$$

This is a hyperbola in V_5 . It is not a closed curve, and two geodesics of this type starting from \mathbf{P} never meet again.

We note that

$$\omega = 1 \text{ for } \begin{cases} K > 0 \text{ and } \Gamma \text{ spacelike,} \\ K < 0 \text{ and } \Gamma \text{ timelike;} \end{cases} \quad (55)$$

$$\omega = -1 \text{ for } \begin{cases} K > 0 \text{ and } \Gamma \text{ timelike,} \\ K < 0 \text{ and } \Gamma \text{ spacelike.} \end{cases} \quad (56)$$

Comparing the results just obtained with those obtained from (16)–(19), we find agreement with regard to the meeting of geodesics. But our new results are stronger. For $K < 0$, the timelike geodesics not only meet as shown in Fig. 3, but they are *closed curves*, and we are compelled to redraw Fig. 3 as in Fig. 4. This depicts what can only be described as a fantastic situation. We see a test particle repeating its history over and over again! This is at variance with our basic ideas

of causality, and we conclude that a de Sitter universe with K negative involves ideas of altogether too revolutionary a character for physics as it exists today.

§ 2. METRIC FORMS FOR SPHERICAL SYMMETRY

To quote WEYL [1952]: 'Symmetry, as wide or as narrow as you may define its meaning, is one idea by which man through the ages has tried to comprehend and create order, beauty, and perfection.'

The concept of symmetry is built into us so deeply that it is hard to explain what it means in words which convey more than the word itself does. One appreciates immediately, for example, the symmetries possessed by the equilateral triangle, the square, or the circle. But intuition has its limitations, and any serious discussion of symmetry leads to the theory of groups. However there is a danger that, in the pursuit of a mathematical formalism, valuable intuitive perceptions may be obscured, and so, in accordance with the geometric spirit of this book, we shall try to preserve in discussing symmetric space-times some of the intuitive spirit of elementary geometry.

It will be convenient to use the word *equivalent* in a sense which is best conveyed by a fantastic example. Suppose a man wishes to bury a box of treasure and leave instructions so that it may be recovered later. The instructions are to involve nothing but geodetic measurements without reference to recognized landmarks. If he lives on a perfect sphere, it is impossible for him to give satisfactory instructions, because every point on a perfect sphere is *equivalent* to every other point. If he lives on an ellipsoid of revolution, his best plan is to bury the box at one of the two poles; true, the two poles are *equivalent* and so his instructions cannot identify the place precisely, but the poles are better than elsewhere, since all points on a parallel of latitude are *equivalent*. If he lives on a pear-shaped planet, the two poles are individually *identifiable* by geodetic measurements carried out in their

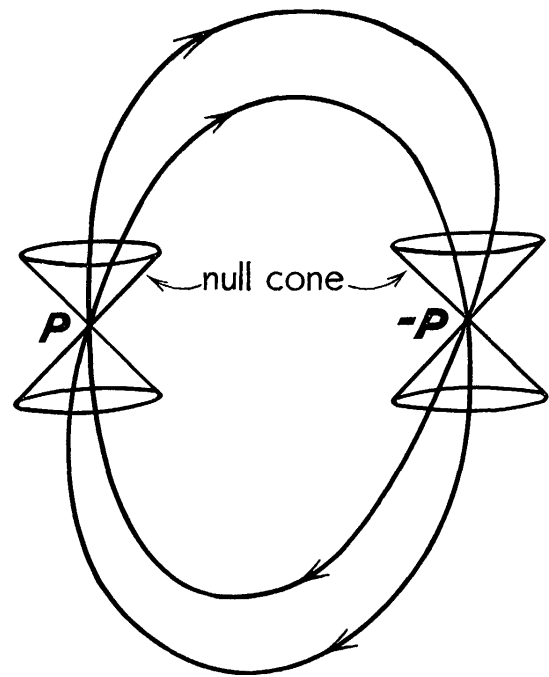


Fig. 4 – Closed timelike geodesics in embedded de Sitter universe with $K < 0$

neighbourhoods, and he might bury his treasure at the pole which has the smaller (or greater) Gaussian curvature.

We now proceed to symmetric space-times, armed with the words *equivalent* and *identifiable*, to be used in the sense indicated above by analogy; it is of course understood that the 'geodetic' measurements to be made in space-time refer to the Riemannian metric.

In the flat Minkowskian space-time of special relativity, all events are equivalent, all future-pointing timelike unit vectors are equivalent, all spacelike unit vectors are equivalent, all future-pointing null vectors are equivalent. No one of the elements just mentioned is identifiable. These statements are true also for the de Sitter universe discussed in the preceding section. In fact, the flat space-time of Minkowski and the de Sitter universe are as symmetric as space-time can be; there remain always the distinctions between past and future, and between timelike, null, and spacelike.

When, in Newtonian physics, we create a simple model for the discussion of the gravitational field of the sun or a pulsating star, we impose spherical symmetry in a sense well understood. Our present task is to transport this concept of spherical symmetry into general relativity in order to discuss the gravitational field of the sun or a pulsating star, the latter including the former as a special case.

The specification of spherical symmetry is simple. We suppose that the world-line C of some particle of the star is an axis of symmetry in the sense that, at each event on C , all unit vectors orthogonal to C are equivalent. But we do not at present assume that all events on C are equivalent.

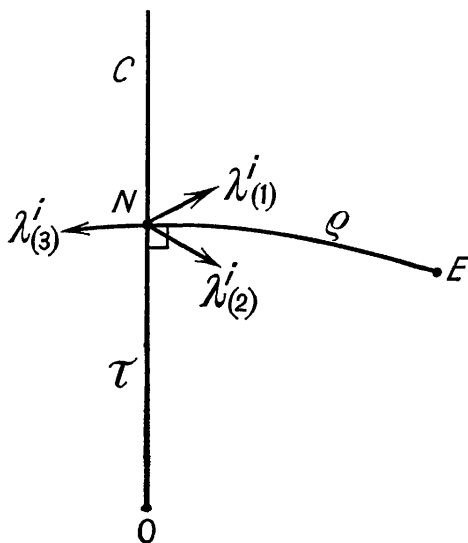


Fig. 5 – Construction for polar Gaussian coordinates

It is evident that C must be a geodesic, since otherwise its first normal would be an identifiable vector orthogonal to C .

Our task is now to calculate the metric tensor g_{ij} in the case of spherical symmetry, but this problem becomes meaningful only after we have specified the coordinates to be used.

There are a number of different coordinate systems, each with its special virtue. We shall start with what we shall call *polar Gaussian coordinates*, defined

as follows. In Fig. 5 we see the central geodesic C with some event O chosen on it. Let $\lambda_{(\alpha)}^i$ be an orthonormal triad chosen orthogonal to C at O , and then carried by parallel transport along C . Let E be any event and EN the geodesic drawn from E to cut C orthogonally. The tangent to NE at N lies in the 3-element of $\lambda_{(\alpha)}^i$ and its direction may be described by the usual polar angles (θ, ϕ) . We write $NE = \rho$, $ON = \tau$. Then $(\rho, \theta, \phi, \tau)$ are our polar Gaussian coordinates. They differ from the Gaussian coordinates of I-§ 8 only in that they are based on the geodesic C instead of on a 3-space. But this difference is rather trivial, and they possess an important property of Gaussian coordinates — they are admissible (except on C). The metric form of space-time is

$$\Phi = d\rho^2 + \Phi_1, \quad (57)$$

where Φ_1 is a quadratic form in $d\theta, d\phi, d\tau$.

By the assumed equivalence of all unit vectors orthogonal to C , Φ must not change if we change the triad $\lambda_{(\alpha)}^i$. A change of this triad is precisely a rotation of axes in Euclidean 3-space, and under such rotation the only invariant differential form in $d\theta, d\phi$ is

$$d\sigma^2 = d\theta^2 + \sin^2 \theta d\phi^2. \quad (58)$$

Accordingly we have, as a general expression for the metric in space-time with spherical symmetry, in terms of polar Gaussian coordinates,

$$\Phi = d\rho^2 + \rho^2 f(\rho, \tau) d\sigma^2 - h(\rho, \tau) d\tau^2, \quad (59)$$

where f and h possess continuous first derivatives. The factor ρ^2 is pulled out of the coefficient of $d\sigma^2$ for purely notational reasons. From the elementary flatness of space-time (the ratio of the circumference of a small circle to its radius is 2π), and from the definition of τ , we have on C

$$f(0, \tau) = 1, \quad h(0, \tau) = 1. \quad (60)$$

We can pass from polar Gaussian coordinates to the *Fermi coordinates* of II-§ 10 (here denoted by x^i) by writing

$$x^1 = \rho \sin \theta \cos \phi, \quad x^2 = \rho \sin \theta \sin \phi, \quad x^3 = \rho \cos \theta, \quad x^4 = \tau. \quad (61)$$

Then (remember that Greek suffixes take the values 1, 2, 3) we have

$$x^\alpha x^\alpha = \rho^2, \quad x^\alpha dx^\alpha = \rho d\rho, \quad dx^\alpha dx^\alpha = d\rho^2 + \rho^2 d\sigma^2, \quad (62)$$

and (59) gives

$$\begin{aligned} \Phi &= g_{ij}dx^i dx^j, \\ g_{\alpha\beta} &= f\delta_{\alpha\beta} + \rho^{-2}(1-f)x^\alpha x^\beta, \quad g_{\alpha 4} = 0, \quad g_{44} = -h, \end{aligned} \quad (63)$$

f and h being functions of ρ and x^4 .

For some curious reason, neither of the above systems of coordinates gives maximum simplicity in the field equations, and we now construct a third system which we call *curvature coordinates*. Consider the 2-space for which (ρ, τ) have fixed values and (θ, ϕ) are current coordinates. By virtue of the assumed spherical symmetry, all points of this 2-space are equivalent. It is therefore a 2-space of constant intrinsic Gaussian curvature, say $1/r^2$, r being a function of (ρ, τ) . This 2-space is in fact intrinsically indistinguishable from an ordinary sphere of radius r : it has an invariant area $4\pi r^2$ (this is perhaps the easiest way of remembering what r is), and the metric on it is

$$ds^2 = r^2 d\sigma^2. \quad (64)$$

Comparison with (59) gives

$$r^2 = \rho^2 f. \quad (65)$$

It is evident that $r = 0$ on C .

We now consider the other type of 2-space, on which (θ, ϕ) have fixed values and (ρ, τ) are current coordinates. In it we draw the curves $r = \text{const.}$ (Fig. 6), and their orthogonal trajectories, such as EM .

Fig. 6 – Construction for curvature coordinates (r, t)

Then any event E defines an event M on C , and, if we write $OM = t$, then we have in (r, t) a system of orthogonal coordinates in the 2-space in question. Thus, in terms of these *curvature coordinates*, we have

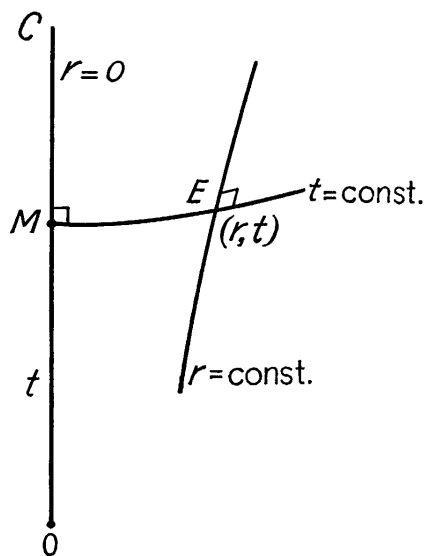
$$\Phi = A(r, t)dr^2 + r^2 d\sigma^2 - B(r, t)dt^2, \quad (66)$$

with

$$A(0, t) = 1, \quad B(0, t) = 1. \quad (67)$$

We use the word *curvature* on account of the way in which r is defined.

But, although these curvature coordinates simplify the field equations as we shall see later, they have a certain disadvantage. The Gaussian coordinates $(\rho, \theta, \phi, \tau)$ are admissible, but, in the process of



obtaining the orthogonal trajectories of the lines $r = \text{const.}$ a degree of smoothness is lost, and we must be prepared for discontinuities in the first derivatives of A and B , although these functions themselves are continuous. This question of smoothness, which has caused considerable confusion, has been carefully examined by ISRAEL [1958]; we shall not discuss it further here.

We pass now to a fourth system of coordinates, *null coordinates*. Taking (θ, ϕ) as before, we define coordinates x^1, x^4 of an event E (Fig. 7) by drawing through E the complete null cone, cutting C at P and Q , say. We write $OP = x^1$, $OQ = x^4$. Then the null coordinates of E are (x^1, θ, ϕ, x^4) and the metric form is (as is easy to see from the null character of PE and EQ).

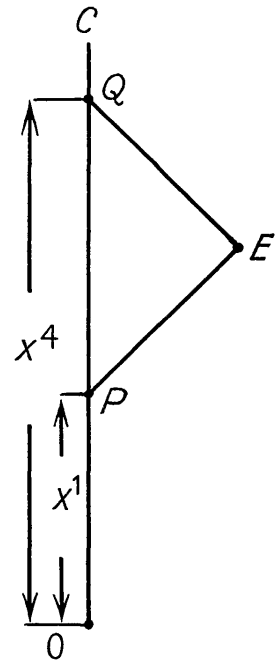


Fig. 7 – Construction for null coordinates

$$\Phi = -2F(x^1, x^4)dx^1dx^4 + H(x^1, x^4)d\sigma^2. \quad (68)$$

Here F and H are arbitrary functions¹, H being the same as r^2 .

We have now obtained in (59), (63), (66) and (68) four different (but of course equivalent) ways of writing the metric form in space-time with spherical symmetry. In each case there are *two* unknown functions of *two* independent variables. Thus the imposition of spherical symmetry gives a great simplification; in a general field, with Gaussian coordinates, there are *six* unknown functions of *four* independent variables. It is because of this simplification that we can really get something done in the case of spherical symmetry.

There are of course other ways of writing the metric form for spherical symmetry. There are *isothermal coordinates* for which

$$\Phi = C(x^1, x^4)[(dx^1)^2 - (dx^4)^2] + H(x^1, x^4)d\sigma^2, \quad (69)$$

and *isotropic coordinates* for which

$$\Phi = M(x^\beta x^\beta, x^4)dx^\alpha dx^\alpha - N(x^\beta x^\beta, x^4)(dx^4)^2. \quad (69a)$$

However there is always a danger in losing sight of the geometrical meaning of the coordinates employed.

¹ For conditions on the axis C , see SYNGE [1957b].

§ 3. VARIOUS FORMULAE FOR SPHERICAL SYMMETRY

Spherical symmetry is so interesting by virtue of its comparative simplicity, and the physical problems associated with it are so far from being exhausted, that it seems wise to develop in some detail a variety of formulae. If we make our calculations for the form ¹

$$\Phi = e^{\alpha}(dx^1)^2 + e^{\beta}[(dx^2)^2 + \sin^2 x^2(dx^3)^2] - e^{\gamma}(dx^4)^2, \quad (70)$$

where α, β, γ are *three* functions of (x^1, x^4) , we can specialize our results by the following special demands:

Polar Gaussian coordinates: $\alpha = 0$.

Curvature coordinates: $\beta = 2 \log x^1 \quad (x^1 = r).$ (71)

Isothermal coordinates: $\alpha = \gamma$.

Isotropic coordinates: $\alpha = \beta$.

This will not cover Fermi coordinates or null coordinates; they must be dealt with separately, if we are to avoid the tedium of transforming from one coordinate system to another.

In working with (70), we shall use the following notation:

$$\sin x^2 = s, \quad \cos x^2 = c. \quad (72)$$

We shall indicate the partial derivatives of α, β, γ with respect to (x^1, x^4) by subscripts without commas, so that, for example,

$$\alpha_1 = \frac{\partial \alpha}{\partial x^1}, \quad \gamma_{14} = \frac{\partial^2 \gamma}{\partial x^1 \partial x^4}. \quad (73)$$

For the form (70) we have

$$\begin{aligned} g_{11} &= e^{\alpha}, & g_{22} &= e^{\beta}, & g_{33} &= e^{\beta} s^2, & g_{44} &= -e^{\gamma}, & g_{ij} &= 0 \quad (i \neq j), \\ g^{11} &= e^{-\alpha}, & g^{22} &= e^{-\beta}, & g^{33} &= e^{-\beta} s^{-2}, & g^{44} &= -e^{-\gamma}, & g^{ij} &= 0 \quad (i \neq j), \\ g &= \det g_{ij} = -s^2 \exp(\alpha + 2\beta + \gamma), \\ \log \sqrt{-g} &= \frac{1}{2}\alpha + \beta + \frac{1}{2}\gamma + \log s. \end{aligned} \quad (74)$$

¹ For a general orthogonal metric, the Christoffel symbols and the Einstein tensor were calculated explicitly by DINGLE [1933a] and will be found in TOLMAN [1934b, p. 254] and McVITTIE [1956, p. 69], but they are naturally rather formidable.

The surviving components of the Christoffel symbol Γ_{jk}^i are as follows:

$$\begin{aligned}
 \Gamma_{11}^1 &= \tfrac{1}{2}\alpha_1, & \Gamma_{11}^4 &= \tfrac{1}{2}\alpha_4 e^{\alpha-\gamma}, \\
 \Gamma_{12}^2 &= \Gamma_{13}^3 = \tfrac{1}{2}\beta_1, \\
 \Gamma_{14}^1 &= \tfrac{1}{2}\alpha_4, & \Gamma_{14}^4 &= \tfrac{1}{2}\gamma_1, \\
 \Gamma_{22}^1 &= -\tfrac{1}{2}\beta_1 e^{\beta-\alpha}, & \Gamma_{22}^4 &= \tfrac{1}{2}\beta_4 e^{\beta-\gamma}, \\
 \Gamma_{23}^3 &= cs^{-1}, & \Gamma_{24}^2 &= \Gamma_{34}^3 = \tfrac{1}{2}\beta_4, \\
 \Gamma_{33}^1 &= -\tfrac{1}{2}\beta_1 e^{\beta-\alpha}s^2, & \Gamma_{33}^2 &= -sc, & \Gamma_{33}^4 &= \tfrac{1}{2}\beta_4 e^{\beta-\gamma}s^2, \\
 \Gamma_{44}^1 &= \tfrac{1}{2}\gamma_1 e^{\gamma-\alpha}, & \Gamma_{44}^4 &= \tfrac{1}{2}\gamma_4.
 \end{aligned} \tag{75}$$

Hence, by direct calculation from (88), the surviving components of the Riemann tensor are found to be as follows:

$$\begin{aligned}
 R_{2323} &= s^2 e^{\beta} (1 - \tfrac{1}{4}\beta_1^2 e^{\beta-\alpha} + \tfrac{1}{4}\beta_4^2 e^{\beta-\gamma}), \\
 R_{1212} &= e^{\beta} (-\tfrac{1}{2}\beta_{11} - \tfrac{1}{4}\beta_1^2 + \tfrac{1}{4}\alpha_1\beta_1) + \tfrac{1}{4}\alpha_4\beta_4 e^{\alpha+\beta-\gamma}, \\
 R_{3131} &= s^2 R_{1212}, \\
 R_{1224} &= e^{\beta} (\tfrac{1}{2}\beta_{14} + \tfrac{1}{4}\beta_1\beta_4 - \tfrac{1}{4}\alpha_4\beta_1 - \tfrac{1}{4}\beta_4\gamma_1), \\
 R_{3134} &= -s^2 R_{1224}, \\
 R_{1414} &= e^{\alpha} (-\tfrac{1}{2}\alpha_{44} - \tfrac{1}{4}\alpha_4^2 + \tfrac{1}{4}\alpha_4\gamma_4) + e^{\gamma} (\tfrac{1}{2}\gamma_{11} + \tfrac{1}{4}\gamma_1^2 - \tfrac{1}{4}\gamma_1\alpha_1), \\
 R_{2424} &= e^{\beta} (-\tfrac{1}{2}\beta_{44} - \tfrac{1}{4}\beta_4^2 + \tfrac{1}{4}\beta_4\gamma_4) + \tfrac{1}{4}\beta_1\gamma_1 e^{\beta-\alpha+\gamma}, \\
 R_{3434} &= s^2 R_{2424}.
 \end{aligned} \tag{76}$$

We note that those components vanish in which there is just one subscript 2 or just one subscript 3, a fact which is easy to verify without calculation on the basis of the symmetry alone. Likewise for the Ricci tensor, a component vanishes if it has just one subscript 2 or one subscript 3, and calculation gives for the surviving components the following values:

$$\begin{aligned}
 R_{11} &= \beta_{11} + \tfrac{1}{2}\beta_1^2 + \tfrac{1}{2}\gamma_{11} + \tfrac{1}{4}\gamma_1^2 - \tfrac{1}{2}\alpha_1\beta_1 - \tfrac{1}{4}\alpha_1\gamma_1 \\
 &\quad + e^{\alpha-\gamma} (-\tfrac{1}{2}\alpha_{44} - \tfrac{1}{4}\alpha_4^2 - \tfrac{1}{2}\alpha_4\beta_4 + \tfrac{1}{4}\alpha_4\gamma_4), \\
 R_{22} &= -1 + e^{\beta-\alpha} (\tfrac{1}{2}\beta_{11} + \tfrac{1}{2}\beta_1^2 + \tfrac{1}{4}\beta_1\gamma_1 - \tfrac{1}{4}\alpha_1\beta_1) \\
 &\quad + e^{\beta-\gamma} (-\tfrac{1}{2}\beta_{44} - \tfrac{1}{2}\beta_4^2 - \tfrac{1}{4}\alpha_4\beta_4 + \tfrac{1}{4}\beta_4\gamma_4), \\
 R_{33} &= s^2 R_{22}, \\
 R_{44} &= \beta_{44} + \tfrac{1}{2}\beta_4^2 + \tfrac{1}{2}\alpha_{44} + \tfrac{1}{4}\alpha_4^2 - \tfrac{1}{4}\alpha_4\gamma_4 - \tfrac{1}{2}\beta_4\gamma_4 \\
 &\quad + e^{\gamma-\alpha} (-\tfrac{1}{2}\gamma_{11} - \tfrac{1}{4}\gamma_1^2 + \tfrac{1}{4}\alpha_1\gamma_1 - \tfrac{1}{2}\beta_1\gamma_1), \\
 R_{14} &= R_{41} = \beta_{14} + \tfrac{1}{2}\beta_1\beta_4 - \tfrac{1}{2}\alpha_4\beta_1 - \tfrac{1}{2}\beta_4\gamma_1.
 \end{aligned} \tag{77}$$

Hence, as the culmination of this calculation, we obtain the following expressions for the surviving components of the mixed Einstein tensor for the form (70):

$$\begin{aligned}
 G_1^1 &= e^{-\alpha}(-\frac{1}{4}\beta_1^2 - \frac{1}{2}\beta_1\gamma_1) + e^{-\beta} + e^{-\gamma}(\beta_{44} + \frac{3}{4}\beta_4^2 - \frac{1}{2}\beta_4\gamma_4), \\
 G_2^2 = G_3^3 &= e^{-\alpha}(-\frac{1}{2}\beta_{11} - \frac{1}{4}\beta_1^2 - \frac{1}{2}\gamma_{11} - \frac{1}{4}\gamma_1^2 - \frac{1}{4}\beta_1\gamma_1 + \frac{1}{4}\alpha_1\beta_1 + \frac{1}{4}\alpha_1\gamma_1) \\
 &\quad + e^{-\gamma}(\frac{1}{2}\beta_{44} + \frac{1}{4}\beta_4^2 + \frac{1}{2}\alpha_{44} + \frac{1}{4}\alpha_4^2 + \frac{1}{4}\alpha_4\beta_4 - \frac{1}{4}\beta_4\gamma_4 - \frac{1}{4}\alpha_4\gamma_4), \quad (78) \\
 G_4^4 &= e^{-\alpha}(-\beta_{11} - \frac{3}{4}\beta_1^2 + \frac{1}{2}\alpha_1\beta_1) + e^{-\beta} + e^{-\gamma}(\frac{1}{4}\beta_4^2 + \frac{1}{2}\alpha_4\beta_4), \\
 e^\alpha G_4^1 &= -e^\gamma G_1^4 = \beta_{14} + \frac{1}{2}\beta_1\beta_4 - \frac{1}{2}\alpha_4\beta_1 - \frac{1}{2}\beta_4\gamma_1.
 \end{aligned}$$

For polar Gaussian coordinates we are to put $\alpha = 0$ in the above formulae, for isothermal coordinates $\alpha = \gamma$, and for isotropic coordinates $\alpha = \beta$. These substitutions produce some simplifications, but we shall not trouble to write out the resulting formulae. For curvature coordinates, however, the simplifications are greater. We have

$$\begin{aligned}
 r &= x^1, \quad \theta = x^2, \quad \phi = x^3, \quad t = x^4 \\
 \Phi &= e^\alpha dr^2 + r^2 d\sigma^2 - e^\gamma dt^2, \\
 d\sigma^2 &= d\theta^2 + \sin^2 \theta d\phi^2, \quad (79) \\
 e^\beta &= r^2, \quad \beta = 2 \log r, \quad \beta_1 = 2r^{-1}, \quad \beta_{11} = -2r^{-2}, \\
 \beta_{11} + \frac{1}{2}\beta_1^2 &= 0, \quad \beta_4 = 0.
 \end{aligned}$$

Then (76) gives ($s = \sin \theta$)

$$\begin{aligned}
 R_{2323} &= s^2 r^{-2} (1 - e^{-\alpha}), \quad R_{1212} = \frac{1}{2} r \alpha_1, \quad R_{3131} = s^2 R_{1212}, \\
 R_{1224} &= -\frac{1}{2} r \alpha_4, \quad R_{3134} = -s^2 R_{1224}, \\
 R_{1414} &= e^\alpha (-\frac{1}{2}\alpha_{44} - \frac{1}{4}\alpha_4^2 + \frac{1}{4}\alpha_4\gamma_4) + e^\gamma (\frac{1}{2}\gamma_{11} + \frac{1}{4}\gamma_1^2 - \frac{1}{4}\alpha_1\gamma_1), \quad (80) \\
 R_{2424} &= \frac{1}{2} r \gamma_1 e^{\gamma-\alpha}, \quad R_{3434} = s^2 R_{2424},
 \end{aligned}$$

and (78) gives

$$\begin{aligned}
 G_1^1 &= r^{-2} - r^{-2} e^{-\alpha} (1 + r \gamma_1), \\
 G_2^2 = G_3^3 &= e^{-\alpha} (-\frac{1}{2}\gamma_{11} - \frac{1}{4}\gamma_1^2 - \frac{1}{2}r^{-1}\gamma_1 + \frac{1}{2}r^{-1}\alpha_1 + \frac{1}{4}\alpha_1\gamma_1) \\
 &\quad + e^{-\gamma} (\frac{1}{2}\alpha_{44} + \frac{1}{4}\alpha_4^2 - \frac{1}{4}\alpha_4\gamma_4), \quad (81) \\
 G_4^4 &= r^{-2} - r^{-2} e^{-\alpha} (1 - r \alpha_1), \\
 e^\alpha G_4^1 &= -e^\gamma G_1^4 = -r^{-1} \alpha_4.
 \end{aligned}$$

Note that no mention has been made here of field equations. The formulae result from the imposition of spherical symmetry on Riemannian space-time, and nothing else. Let us push the work a little further in this purely geometrical spirit.

Except for the conditions (67) for elementary flatness, which now read

$$\alpha = \gamma = 0 \text{ for } r = 0, \quad (82)$$

α and γ are arbitrary functions of (r, t) or (x^1, x^4) , and (81) gives the corresponding Einstein tensor by differentiation. However these formulae have a remarkable property — we can solve for (α, γ) in terms of G_1^1 and G_4^4 quite simply. From the third equation, with (82), we get

$$e^{-\alpha} = 1 - \frac{1}{r} \int_0^r r^2 G_4^4 dr. \quad (83)$$

With α thus obtained, the first of (81) with (82) gives

$$\gamma = \int_0^r \left(\frac{e^\alpha - 1}{r} - r e^\alpha G_1^1 \right) dr. \quad (84)$$

We can then use the other equations of (81) to express the other components of G_j^i in terms of G_1^1 and G_4^4 . By the last of (81) we have

$$G_4^1 = -e^{\gamma-\alpha} G_1^4 = -\frac{1}{r^2} \int_0^r r^2 G_{4,4}^4 dr. \quad (85)$$

But to evaluate G_2^2 we can use, instead of (81), the identity

$$G_{i|j}^j = 0 \quad (86)$$

with $i = 1$; this gives

$$G_2^2 = G_3^3 = \frac{1}{2} r G_{1,1}^1 + \frac{1}{2} r G_{1,4}^4 + (1 + \frac{1}{4} r \gamma_1) G_1^1 + \frac{1}{4} r (\alpha_4 + \gamma_4) G_1^4 - \frac{1}{4} r \gamma_1 G_4^4. \quad (87)$$

The significance of all this is that we have essentially *two* arbitrary functions, (α, γ) or (G_1^1, G_4^4) , and in terms of these the other G_j^i are determined by differentiation or integration.

As indicated earlier, curvature coordinates are not admissible in the

technical sense. Thus across a 3-space of discontinuity with equation

$$f(r, t) = 0 \quad (88)$$

we are to assume the continuity of α and γ , but not necessarily the continuity of their first derivatives. The junction conditions are in fact, as in I-(229),

$$G_1^1 f_{,1} + G_1^4 f_{,4} = [C], \quad G_4^1 f_{,1} + G_4^4 f_{,4} = [C], \quad (89)$$

where $[C]$ means continuous. In the particular case where the discontinuity is $r = \text{const.}$, these conditions reduce to

$$G_1^1 = [C], \quad G_4^1 = [C]. \quad (90)$$

§ 4. THE EXTERIOR SCHWARZSCHILD FIELD

Consider a star or other spherically symmetric distribution of matter. At present we are not interested in its interior, but only in a domain $r > a$ (we use curvature coordinates) in which there is no matter. In that domain we have $T_{ij} = 0$, and so the field equations read

$$G_j^i - \Lambda \delta_j^i = 0; \quad (91)$$

we include the cosmological constant for the sake of generality. But in view of the spherical symmetry we have to consider only the equations

$$\begin{aligned} G_1^1 &= \Lambda, \\ G_4^4 &= \Lambda, \\ G_4^1 &= 0. \end{aligned} \quad (92)$$

The other equations

$$G_2^2 = \Lambda, \quad G_3^3 = \Lambda, \quad (93)$$

will then be satisfied by virtue of the identity (87).

Substituting in (92) from (81), we have the three equations

$$\begin{aligned} e^{-\alpha}(1 + r\gamma_1) &= 1 - \Lambda r^2, \\ e^{-\alpha}(1 - r\alpha_1) &= 1 - \Lambda r^2, \\ \alpha_4 &= 0. \end{aligned} \quad (94)$$

From the last of these we have

$$\alpha = \alpha(r), \quad (95)$$

and then from the second

$$e^{-\alpha} = 1 - \frac{A}{r} - \frac{1}{3}Ar^2, \quad (96)$$

where A is an arbitrary constant. (We must not apply (82) here, because the domain $r > a$ under discussion does not include the axis of symmetry $r = 0$). Subtracting the second of (94) from the first, we get

$$\alpha_1 + \gamma_1 = 0, \quad (97)$$

and so

$$\gamma = -\alpha + F(t), \quad (98)$$

where F is an arbitrary function. Hence the metric for the domain $r > a$ is

$$\Phi = \frac{dr^2}{1 - \frac{A}{r} - \frac{1}{3}Ar^2} + r^2 d\sigma^2 - \left(1 - \frac{A}{r} - \frac{1}{3}Ar^2\right) e^{F(t)} dt^2. \quad (99)$$

If we change from t to t' by the transformation

$$t' = \int \exp\left[\frac{1}{2}F(t)\right] dt, \quad (100)$$

we get

$$\Phi = \frac{dr^2}{1 - \frac{A}{r} - \frac{1}{3}Ar^2} + r^2 d\sigma^2 - \left(1 - \frac{A}{r} - \frac{1}{3}Ar^2\right) dt'^2. \quad (101)$$

This may be referred to as the *Schwarzschild exterior field* [SCHWARZSCHILD, 1916a], although the term is properly used only if we put $A = 0$.

A space-time is called *stationary* if coordinates exist so that

$$g_{ij,4} = 0. \quad (102)$$

A stationary space-time admits a group of motions (cf. VI-§ 3). If in addition to (102) we have

$$g_{\alpha 4} = 0, \quad (103)$$

so that the metric form is

$$\Phi = g_{\alpha\beta}dx^\alpha dx^\beta + g_{44}(dx^4)^2, \quad (104)$$

the space-time is said to be *static*.

It is clear that space-time with the metric form (101) is static. In fact, *any spherically symmetric field in vacuo is static*. This remarkable result is often referred to as 'Birkhoff's theorem' ¹.

Some critical remarks may be made about the formula (101).

First, in § 2 the coordinate t was given a precise definition as proper time on the axis $r = 0$. In passing to t' by (100) this meaning is lost. However, if we drop the embarrassing constant A (see below), so that (101) becomes

$$\Phi = \frac{dr^2}{1 - \frac{A}{r}} + r^2 d\sigma^2 - \left(1 - \frac{A}{r}\right) dt'^2, \quad (105)$$

and let r tend to infinity, we see that dt' is the element of proper time for a particle which is fixed in the sense that (r, θ, ϕ) are constant.

Secondly, to preserve the signature of Φ in (101), we must have

$$1 - \frac{A}{r} - \frac{1}{3}Ar^2 > 0. \quad (106)$$

If we assume A positive, as is usually done, it is clear that this inequality will be broken for sufficiently large values of r . However, this disaster is forestalled by another. As will be shown below, r attains a maximum as we go out along the geodesics NE of Fig. 5, and then decreases. This means that, beyond the maximum, the curvature coordinates cannot be used because there is not a 1 : 1 correspondence between events and coordinate tetrads (r, θ, ϕ, t) .

To examine the behaviour of r , we use polar Gaussian coordinates, so that the metric form reads ²

$$\Phi = d\rho^2 + r^2 d\sigma^2 - e^\gamma d\tau^2. \quad (107)$$

To agree with the notation of § 3, we write

$$\rho = x^1, \quad \tau = x^4, \quad r^2 = e^\beta, \quad (108)$$

and remember that now r is not an independent variable but a function

¹ Cf. JEBSEN [1921], ALEXANDROW [1923], BIRKHOFF [1923].

² It is assumed that there is no intersection for $\rho > 0$ of neighbouring geodesics NE of Fig. 5; equivalently, $r > 0$, $e^\gamma > 0$ for $\rho > 0$.

of (x^1, x^4) . Of the field equations in vacuo, we shall use only

$$G_4^4 = \Lambda, \quad G_4^1 = 0. \quad (109)$$

Since we are using polar Gaussian coordinates, we put $\alpha = 0$ in (78), and the equations (109) read

$$\begin{aligned} -\beta_{11} - \frac{3}{4}\beta_1^2 + e^{-\beta} + \frac{1}{4}e^{-\gamma}\beta_4^2 &= \Lambda, \\ \beta_{14} + \frac{1}{2}\beta_1\beta_4 - \frac{1}{2}\beta_4\gamma_1 &= 0. \end{aligned} \quad (110)$$

Now

$$\begin{aligned} \beta &= 2 \log r, & \beta_1 &= \frac{2r_1}{r}, & \beta_4 &= \frac{2r_4}{r}, \\ \beta_{11} &= \frac{2r_{11}}{r} - \frac{2r_1^2}{r^2}, & \beta_{14} &= \frac{2r_{14}}{r} - \frac{2r_1r_4}{r^2}, \end{aligned} \quad (111)$$

and so (110) become

$$\begin{aligned} 1 - 2rr_{11} - r_1^2 + e^{-\gamma}r_4^2 &= \Lambda r^2, \\ 2r_{14} - r_4\gamma_1 &= 0. \end{aligned} \quad (112)$$

The second of these gives

$$r_4 e^{-\frac{1}{2}\gamma} = B(x^4), \quad (113)$$

where the function B is arbitrary. The first of (112) then gives

$$1 - 2rr_{11} - r_1^2 + B^2 = \Lambda r^2, \quad (114)$$

which may be written

$$(rr_1^2)_1 = (1 - \Lambda r^2 + B^2)r_1, \quad (115)$$

and hence

$$r_1^2 = 1 - \frac{1}{3}\Lambda r^2 + B^2 - \frac{C}{r}, \quad (116)$$

where C is another arbitrary function of x^4 . This last equation may, for fixed x^4 , be integrated by a quadrature, and inversion will then give the function $r(x^1, x^4)$. However it is evident from (116) that, if Λ is positive, r cannot increase indefinitely as x^1 increases; it will have a maximum, and then decrease, as indicated above ¹.

¹ Questions of this nature have been studied in some detail by O'RAIFEARTAIGH [1958a].

§ 5. THE COMPLETE FIELD OF A SPHERICALLY SYMMETRIC DISTRIBUTION OF MATTER

In the preceding section we considered the exterior field of a spherically symmetric distribution of matter; it seemed desirable to deal separately with the exterior field in order to bring out the fact that Birkhoff's theorem is independent of the structure of the matter, provided of course that it is spherically symmetric. Now we turn to the general problem of the total field, exterior and interior.

To fix our ideas, we think of a star, which may pulsate radially¹. For the sake of formal simplicity, we shall use curvature coordinates, so that the metric form is

$$\begin{aligned}\Phi &= e^\alpha dr^2 + r^2 d\sigma^2 - e^\gamma dt^2, \\ d\sigma^2 &= d\theta^2 + \sin^2 \theta d\phi^2, \\ x^1 &= r, \quad x^2 = \theta, \quad x^3 = \phi, \quad x^4 = t.\end{aligned}\tag{117}$$

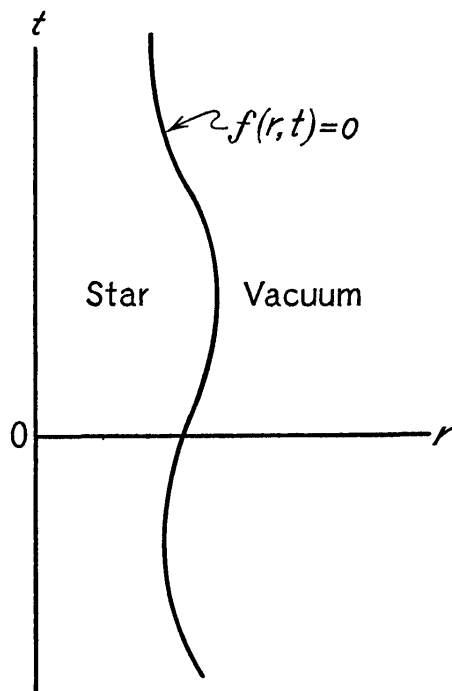


Fig. 8 – Space-time picture of an equation pulsating star

For generality we shall include the cosmological constant Λ , so that the field equations read

$$G_j^i - \Lambda \delta_j^i = -\kappa T_j^i, \quad \kappa = 8\pi. \tag{118}$$

But, in view of the embarrassments indicated in the preceding section, we shall put $\Lambda = 0$ on occasion.

It is really a matter of indifference whether we suppose the star terminated by a sharp boundary, or extending diffusely to infinity. We can regard the former as a limiting case of the latter. If we prefer the sharp boundary, we represent it by

$$f(r, t) = 0, \tag{119}$$

and show it diagrammatically as in Fig. 8. Across the sharp boundary we have to satisfy the junction conditions (89).

¹ A superficial interpretation of Birkhoff's theorem might suggest that, since the exterior field is static, a star cannot pulsate! That would be quite a false conclusion. A star can indeed pulsate with spherical symmetry, but in relativity, as in Newtonian theory, these pulsations do not affect the exterior gravitational field. There is, in fact, no gravitational 'monopole radiation'.

Spherical symmetry imposes restrictions on the energy tensor. Its eigenvectors λ^i and eigenvalues k are such that, at any event, two eigenvectors lie in the 2-element for which $dx^2 = dx^3 = 0$ and the other two in the 2-element for which $dx^1 = dx^4 = 0$. In the latter case the two eigenvalues are equal and the two eigenvectors indeterminate. The equations for the eigenvectors and eigenvalues are

$$T_{ij}\lambda^j = kg_{ij}\lambda^j. \quad (120)$$

Remembering that g_{ij} is diagonal, we see that the equations pertaining to $dx^1 = dx^4 = 0$ read

$$\begin{aligned} T_{12}\lambda^2 + T_{13}\lambda^3 &= 0, \\ T_{22}\lambda^2 + T_{23}\lambda^3 &= kg_{22}\lambda^2, \\ T_{32}\lambda^2 + T_{33}\lambda^3 &= kg_{33}\lambda^3, \\ T_{42}\lambda^2 + T_{43}\lambda^3 &= 0. \end{aligned} \quad (121)$$

These are to be satisfied by some k with the ratio $\lambda^2 : \lambda^3$ arbitrary. Hence

$$\begin{aligned} T_{12} = T_{13} = T_{42} = T_{43} = T_{23} &= 0, \\ T_{22}/g_{22} &= T_{33}/g_{33}, \end{aligned} \quad (122)$$

and so the only surviving components of T_j^i are

$$T_1^1, \quad T_2^2 = T_3^3, \quad T_4^4, \quad T_4^1, \quad T_1^4, \quad (123)$$

with

$$e^\alpha T_4^1 = -e^\gamma T_1^4. \quad (124)$$

Of the field equations (118) there are just four to satisfy:

$$\begin{aligned} G_1^1 &= \Lambda - \kappa T_1^1, \\ G_2^2 &= \Lambda - \kappa T_2^2, \\ G_4^4 &= \Lambda - \kappa T_4^4, \\ G_4^1 &= -\kappa T_4^1. \end{aligned} \quad (125)$$

When we substitute for the left hand sides from (81), we have before us four equations connecting the six quantities

$$\alpha, \quad \gamma, \quad T_1^1, \quad T_2^2, \quad T_4^4, \quad T_4^1. \quad (126)$$

Our plan of campaign is to regard T_1^1 and T_4^4 as assigned functions of (r, t) . Instead of solving (125) afresh, we turn to results established in § 3. Substitution from (125) in (83) and (84) gives

$$\begin{aligned} e^{-\alpha} &= 1 - \frac{1}{3}\Lambda r^2 + \frac{\kappa}{r} \int_0^r r^2 T_4^4 dr, \\ \gamma &= -\Lambda \int_0^r r e^{\alpha} dr + \int_0^r \left(\frac{e^{\alpha} - 1}{r} + \kappa r e^{\alpha} T_1^1 \right) dr. \end{aligned} \quad (127)$$

From the first and third of (81) we have

$$G_1^1 - G_4^4 = -r^{-1}e^{-\alpha}(\alpha_1 + \gamma_1), \quad (128)$$

and hence by (125)

$$\alpha_1 + \gamma_1 = \kappa r e^{\alpha} (T_1^1 - T_4^4). \quad (129)$$

Thus we can write the second of (127) in the alternative form

$$\gamma = -\alpha + \kappa \int_0^r r e^{\alpha} (T_1^1 - T_4^4) dr. \quad (130)$$

Having thus expressed α and γ in terms of T_1^1 and T_4^4 , we get the following expressions for the other members of (126) by substituting from (125) in (85) and (87):

$$\begin{aligned} T_4^1 &= -r^{-2} \int_0^r r^2 T_{4,4}^4 dr, \\ T_2^2 &= \frac{1}{2} r T_{1,1}^1 + \frac{1}{2} r T_{1,4}^4 + (1 + \frac{1}{4} r \gamma_1) T_1^1 \\ &\quad - \frac{1}{4} r e^{\alpha - \gamma} (\alpha_4 + \gamma_4) T_4^1 - \frac{1}{4} r \gamma_1 T_4^4. \end{aligned} \quad (131)$$

If we put

$$T_1^1 = 0, \quad T_4^4 = 0, \quad (132)$$

we destroy all the components T_j^i and, in fact, annihilate the star. Then (127) and (130) give

$$e^{-\alpha} = e^{\gamma} = 1 - \frac{1}{3}\Lambda r^2, \quad (133)$$

and the metric form becomes

$$\Phi = \frac{dr^2}{1 - \frac{1}{3}\Lambda r^2} + r^2 d\sigma^2 - (1 - \frac{1}{3}\Lambda r^2) dt^2. \quad (134)$$

If $\Lambda = 0$, this is the metric of flat space-time. If $\Lambda \neq 0$, it is the metric of space-time of constant curvature, $K = \frac{1}{3}\Lambda$. We have in fact rediscovered the de Sitter universe of § 1!

Leaving this very special case, let us put $\Lambda = 0$ and summarize the situation as follows: *Given T_1^1 and T_4^4 arbitrarily as functions of (r, t) , the metric form (117) is consistent with the field equations provided α and γ are given by*

$$\begin{aligned} e^{-\alpha} &= 1 + \frac{\kappa}{r} \int_0^r r^2 T_4^4 dr, \\ \gamma &= \int_0^r \left(\frac{e^\alpha - 1}{r} + \kappa r e^\alpha T_1^1 \right) dr \\ &= -\alpha + \kappa \int_0^r r e^\alpha (T_1^1 - T_4^4) dr, \end{aligned} \quad (135)$$

the remaining components of T_j^i being given by (131).

§ 6. THE MASS OF A BOUNDED STAR AND THE THEOREM OF GAUSS

Let us now consider the case where the star has a sharp boundary as in Fig. 8, with vacuum outside. All that has been said above holds good, but we have to take into account the junction conditions (89), which are equivalent to

$$T_1^1 f_{,1} + T_1^4 f_{,4} = 0, \quad T_4^1 f_{,1} + T_4^4 f_{,4} = 0 \quad \text{for } f = 0. \quad (136)$$

These imply

$$T_1^1 T_4^4 - T_4^1 T_1^4 = 0 \quad \text{for } f = 0, \quad (137)$$

or, by (124) and (131),

$$T_1^1 T_4^4 + e^{\alpha-\gamma} r^{-4} \left(\int_0^r r^2 T_{4,4}^4 dr \right)^2 = 0 \quad \text{for } f = 0. \quad (138)$$

In attempting to construct a model of a star with a sharp boundary, we cannot simply assign T_1^1 and T_4^4 smoothly and then wipe them out beyond some chosen curve $f(r, t) = 0$. The only possible curve is that given by (138), and there is a condition which restricts the original choice of T_1^1 and T_4^4 : if the expression in (138) is denoted by f , then f must satisfy one of (136) on $f = 0$.

Pursuing the case of a bounded spherical star, in general pulsating, and taking $A = 0$, outside the star we have as in (96)

$$e^{-\alpha} = 1 - \frac{A}{r}, \quad (139)$$

where A is a *constant*. On the other hand, if the equation of the boundary of the star is $r = \chi(t)$, (135) gives at any event outside the star

$$e^{-\alpha} = 1 + \frac{\kappa}{r} \int_0^{\chi(t)} r^2 T_4^4 dr. \quad (140)$$

It follows that

$$A = - \kappa \int_0^{\chi(t)} r^2 T_4^4 dr. \quad (141)$$

Thus the right hand side, which we might expect to be a function of t , is actually a constant. This rather surprising fact may be verified as follows. With the aid of (131) we have

$$\frac{d}{dt} \int_0^{\chi(t)} r^2 T_4^4 dr = \int_0^{\chi(t)} r^2 T_{4,4}^4 dr + r^2 T_4^4 \chi'(t) = -r^2 (T_4^1 - T_4^4 \chi'(t)), \quad (142)$$

where T_4^1 and T_4^4 in this last expression are evaluated on the boundary. This vanishes on account of the second junction condition (136), and so the result is verified.

Introducing a numerical factor in order to obtain later an approximate agreement with Newtonian theory [cf. (154) and § 8], we define the *mass* m of a bounded star by

$$m = \frac{1}{2}A = - \frac{1}{2}\kappa \int_0^{\chi(t)} r^2 T_4^4 dr, \quad \kappa = 8\pi. \quad (143)$$

Then for the exterior field we have

$$e^{-\alpha} = 1 - \frac{2m}{r},$$

$$e^{\gamma} = \left(1 - \frac{2m}{r}\right) \exp \left[\kappa \int_0^{x(t)} \frac{T_1^1 - T_4^4}{r - 2m} r^2 dr \right], \quad (144)$$

$$\Phi = \frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\sigma^2 - e^{\gamma} dt^2.$$

In this work t is a well-defined coordinate — it is proper time measured at the star's centre. We already know it is always possible to make the exterior form static. Now we see what the 'time' t' of (105) means. For we get

$$\Phi = \frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\sigma^2 - \left(1 - \frac{2m}{r}\right) dt'^2 \quad (145)$$

by putting

$$t' = \int \exp \left(\frac{1}{2} \kappa \int_0^{x(t)} \frac{T_1^1 - T_4^4}{r - 2m} r^2 dr \right) dt. \quad (146)$$

In the history of relativity, the exterior form (145) has been given priority over the internal form, being regarded as in some way more fundamental. Now it is clear that something goes wrong if $r = 2m$. This is the so-called 'Schwarzschild singularity', and a good deal of consideration has been given to it. However, if we escape from mere formalism and inquire into what we are really doing, we shall find that our work has actually involved some tacit assumptions in the nature of inequalities. It is in fact assumed that the magnitude of T_4^4 is such that $\exp(-\alpha)$ as in (135) is positive everywhere, and this means that events where $r = 2m$ occur *inside* the star. Now (145) applies only *outside* the star, and there is no singularity of (145) *in the domain of its validity*.

We now pass to the theorem of Gauss for a bounded star with spherical symmetry. We recall that, in Newtonian theory, this theorem reads

$$\int N dS = 4\pi m, \quad (147)$$

where N is the inward normal component of gravitational intensity on a surface S enclosing a total mass m .

To carry this over into relativity, we have to find a suitable analogue for Newtonian gravitational intensity. Now if an observer pursuing a t -line (with r, θ, ϕ constant) lets fall a test particle, its geodesic path will deviate from the t -line, and this deviation may be regarded as a measure of gravitational intensity. According to the investigation of the falling apple in III-§ 9 it would seem that we should measure the gravitational intensity by the magnitude b of the first curvature of the t -line, but, as we shall see, the theorem of Gauss comes out best if we use, not the magnitude b , but the first component b^1 of the first curvature vector¹. Since the coordinate system is well-defined geometrically, there is no objection to using the component of a vector, because this component is really an invariant.

The first curvature vector of any curve is

$$b^i = \frac{d^2 x^i}{ds^2} + \Gamma_{jk}^i \frac{dx^j}{ds} \frac{dx^k}{ds}, \quad (148)$$

and for a t -line we have

$$b^1 = \Gamma_{44}^1 \left(\frac{dt}{ds} \right)^2 = e^{-\gamma} \Gamma_{44}^1, \quad (149)$$

or by (75)

$$b^1 = \frac{1}{2} \gamma_1 e^{-\alpha}. \quad (150)$$

At an event outside the star, (135) gives

$$\gamma = \int_0^r \frac{e^\alpha - 1}{r} dr + \kappa \int_0^{x(t)} r e^\alpha T_1^1 dr, \quad (151)$$

$$\gamma_1 = \frac{e^\alpha - 1}{r}, \quad e^{-\alpha} \gamma_1 = \frac{1}{r} (1 - e^{-\alpha}) = \frac{2m}{r^2},$$

and so the gravitational intensity is

$$b^1 = \frac{m}{r^2}; \quad (152)$$

we recognize the inverse square law.

¹ This is the r -component of b^i . We recall the invariant definition of r given in § 2.

Now integrate this intensity over the 2-sphere for which $r = \text{const.}$, $t = \text{const.}$ The element of area is

$$dS = r^2 \sin^2 \theta d\theta d\phi, \quad (153)$$

and so we get

$$\int b^1 dS = 4\pi m, \quad (154)$$

which, on comparison with (147), we recognize as the *theorem of Gauss*.

§ 7. THE FIELD OF A FLUID WITH SPHERICAL SYMMETRY AND THE COMPLETE SCHWARZSCHILD FIELD

In the preceding work we have regarded T_1^1 and T_4^4 as basic functions, arbitrarily assigned except for the condition stated after (138). But this is hardly realistic physically, for matter has some structure, solid or fluid. We shall now find the field of a spherically symmetric distribution of perfect fluid, and in particular the field of a sphere composed of perfect fluid. We shall take $\Lambda = 0$ and consider only the static problem.

As in iv-(84) the energy tensor for a perfect fluid is

$$T_{ij} = (\mu + p)V_i V_j + p g_{ij}, \quad (155)$$

where μ is density, p is pressure, and V^i is 4-velocity, satisfying

$$g_{ij} V^i V^j = -1. \quad (156)$$

In the static problem we take V^i pointing in the t -direction, so that with the metric (117)

$$V^\alpha = 0, \quad V_\alpha = 0, \quad V^4 = e^{-\frac{1}{2}\gamma}, \quad V_4 = -e^{\frac{1}{2}\gamma}. \quad (157)$$

Thus

$$T_1^1 = T_2^2 = T_3^3 = p, \quad T_4^1 = T_1^4 = 0, \quad T_4^4 = -\mu, \quad (158)$$

μ and p being functions of r only.

The field equations are equivalent to (135) with (131). Instead of the second of (135) it is convenient to use (129). Thus, since the first of (131) is identically satisfied, we have to satisfy the following three equations:

$$e^{-\alpha} = 1 - \frac{\kappa}{r} \int_0^r r^2 \mu dr, \quad (159)$$

$$\alpha_1 + \gamma_1 = \kappa r e^\alpha (p + \mu),$$

$$p_1 + \frac{1}{2}\gamma_1(p + \mu) = 0.$$

We recall that the subscript 1 means d/dr .

The density μ may be discontinuous, but α , γ , p are continuous, the continuity of p being required by the junction conditions (90). If the fluid occupies a sphere of radius $r = a$, with vacuum outside, then

$$\mu = 0, \quad p = 0 \text{ for } r > a, \quad (160)$$

and, as we approach $r = a$ from inside, $p \rightarrow 0$.

Since we have only the three equations (159) for the four quantities α , γ , μ , p , it is clear that we have before us no determinate problem. Determinacy is sometimes introduced in fluid problems by assuming a density-pressure relationship, but we shall not do that here. Instead we shall regard the function $\mu(r)$ as assigned. This is of course only a psychological dodge to motivate the work, which consists of statements which are true on the basis of the equations (159) alone.

We shall assume the function $\mu(r)$ smooth to start with, and deal with the case of a sharp boundary later.

The function $\mu(r)$ being given, the first of (159) gives $\alpha(r)$. Eliminating γ from the other two equations, we get the following differential equation for $(p + \mu)$:

$$(p + \mu)_1 + \frac{1}{2}(p + \mu)[\kappa r e^\alpha (p + \mu) - \alpha_1] - \mu_1 = 0. \quad (161)$$

If we define σ by

$$\sigma^{-1} = p + \mu, \quad (162)$$

this equation becomes

$$\sigma_1 + \mu_1 \sigma^2 + \frac{1}{2} \alpha_1 \sigma - \frac{1}{2} \kappa r e^\alpha = 0. \quad (163)$$

We need not trouble to seek an explicit solution. For present purposes it is enough to note that, if the value of p is assigned for some value of r , this equation determines the function $\sigma(r)$ and hence $p(r)$. Then the second of (159) gives γ in the form

$$\gamma = -\alpha + \kappa \int_0^r r e^\alpha \sigma^{-1} dr. \quad (164)$$

The problem of the spherically symmetric fluid is thus solved, at least in principle, with the function $\mu(r)$ arbitrarily assigned.

The equation (163) may be written

$$(\sigma e^{\frac{1}{2}\alpha})_1 = \frac{1}{2} \kappa r e^{3\alpha/2} - \mu_1 \sigma^2 e^{\frac{1}{2}\alpha}, \quad (165)$$

and hence

$$\sigma e^{\frac{1}{2}\alpha} = \sigma_0 + \frac{1}{2} \kappa \int_0^r r e^{3\alpha/2} dr - \int_0^r \mu_1 \sigma^2 e^{\frac{1}{2}\alpha} dr, \quad (166)$$

where σ_0 is the value of σ for $r = 0$. This formula is not, of course, the solution of (163), for the unknown σ is present in the last integral, but it may be useful for iterative processes.

We now pass to the case of a *homogeneous sphere* of fluid, writing

$$\mu = \mu_0 \text{ for } r < a, \quad \mu = 0 \text{ for } r > a, \quad (167)$$

μ_0 being the constant density of the sphere. Taking $p = 0$ for $r > a$ as in (160), we have here a determinate problem, as we shall see.

We deal first with the interior ($r < a$). By the first of (159),

$$e^{-\alpha} = 1 - qr^2, \quad q = \frac{1}{3}\kappa\mu_0 = \frac{8}{3}\pi\mu_0. \quad (168)$$

By (166)

$$\sigma e^{\frac{1}{2}\alpha} = \sigma_0 + \frac{1}{2}\kappa \int_0^r r e^{3\alpha/2} dr; \quad (169)$$

this integral is easy to evaluate, since by (168)

$$e^{-\alpha} d\alpha = 2qr dr, \quad (170)$$

and so we get

$$\sigma = (p + \mu_0)^{-1} = \sigma_0 e^{-\frac{1}{2}\alpha} + \frac{1}{2}\kappa q^{-1}(1 - e^{-\frac{1}{2}\alpha}). \quad (171)$$

The constant σ_0 is evaluated by going to the surface of the sphere, where $r = a$, $p = 0$. When this value is substituted in (171), we get

$$\sigma\mu_0 = \frac{3}{2} - \frac{1}{2} \left(\frac{1 - qr^2}{1 - qa^2} \right)^{\frac{1}{2}}, \quad (172)$$

and so the pressure in the sphere is

$$p = \mu_0 \frac{(1 - qr^2)^{\frac{1}{2}} - (1 - qa^2)^{\frac{1}{2}}}{3(1 - qa^2)^{\frac{1}{2}} - (1 - qr^2)^{\frac{1}{2}}}. \quad (173)$$

The interior solution is completed by finding γ from (164). This gives

$$e^{\gamma} = \left(\frac{3\sqrt{1 - qa^2} - \sqrt{1 - qr^2}}{3\sqrt{1 - qa^2} - 1} \right)^2, \quad (174)$$

and the metric form inside the sphere is

$$\Phi = \frac{dr^2}{1 - qr^2} + r^2 d\sigma^2 - e^{\gamma} dt^2, \quad q = \frac{1}{3}\kappa\mu_0. \quad (175)$$

We recall that t is proper time at the centre of the sphere, and we check that $\gamma = 0$ for $r = 0$, as should be the case.

In accordance with the general definition (143) of the mass of any bounded spherically symmetric distribution of matter, the mass of the fluid sphere is

$$m = \frac{1}{2}\kappa \int_0^a \mu r^2 dr = \frac{1}{6}\kappa\mu_0 a^3 = \frac{4}{3}\pi\mu_0 a^3. \quad (176)$$

Thus, passing to the exterior domain ($r > a$), (159) gives

$$e^{-\alpha} = 1 - \frac{2m}{r}, \quad \gamma = -\alpha + \log C, \quad (177)$$

where C is a constant. By continuity at $r = a$, we have

$$C = (e^{\gamma+\alpha})_{r=a}, \quad (178)$$

with the interior values (168) and (174) inserted, and so

$$C = 4(3\sqrt{1 - qa^2} - 1)^{-2}. \quad (179)$$

Thus the metric form outside the sphere is

$$\Phi = \frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\sigma^2 - \frac{4\left(1 - \frac{2m}{r}\right) dt^2}{(3\sqrt{1 - qa^2} - 1)^2}, \quad q = \frac{1}{3}\kappa\mu_0. \quad (180)$$

We check this for continuity with (175) by noting that

$$\frac{2m}{a} = \frac{1}{3}\kappa\mu_0 a^2 = qa^2. \quad (181)$$

There is a great advantage in using coordinates which are simply defined, physically or geometrically, and t has been used consistently to denote the proper time at the centre of spherical symmetry. However the formulae for a fluid sphere become a little simpler if we change to the time-coordinate

$$t' = 2t(3\sqrt{1 - qa^2} - 1)^{-1}. \quad (182)$$

Then we get from (175) and (180) the complete Schwarzschild field ¹ for

¹ SCHWARZSCHILD [1916a, b]. To transform the exterior form to Gaussian polars, put $d\rho = dr(1 - 2m/r)^{-\frac{1}{2}}$. See VIII-(179) for the exterior form in isotropic coordinates.

a fluid sphere of constant density:

Interior ($r < a$):

$$\Phi_i = \frac{dr^2}{1 - qr^2} + r^2 d\sigma^2 - \left(\frac{3}{2} \sqrt{1 - qa^2} - \frac{1}{2} \sqrt{1 - qr^2} \right)^2 dt'^2; \quad (183)$$

$$\text{Exterior } (r > a): \Phi_e = \frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\sigma^2 - \left(1 - \frac{2m}{r} \right) dt'^2.$$

It is clear that t' is in fact the proper time for a particle fixed at $r = \infty$.

§ 8. ORBITS AND RAYS IN THE SOLAR FIELD

The basic concepts of Einstein's general theory of relativity are so different from those of Newton that one might well be surprised that the physical predictions of the two theories should agree closely¹. But they do, at least under the conditions encountered in nature, for we have to deal only with weak gravitational fields and small relative velocities (the terms 'weak' and 'small' are of course used in a technical sense). This agreement, although it has been indicated earlier in the book, will be demonstrated in the work which follows. It represents a great triumph to have constructed, without using the obnoxious idea of absolute time, a theory of gravitation as successful as Newton's theory has proved from the standpoint of astronomical prediction.

However the two theories do not quite agree in their predictions. Great interest has attached to the small differences between them and doubt has been expressed as to whether observation does in fact completely confirm the predictions of Einstein's theory. A sound judgment in these matters demands experience in observational techniques, and it would be foreign to the spirit of this book to include any dogmatic pronouncement as to whether the said predictions are verified or not². But one thing should be made clear. The issue is not between Newton and Einstein. The concept of absolute time is quite

¹ In this book we are concerned solely with Einstein's theory, but it is necessary to bring in Newtonian theory because astronomers report their observations in Newtonian terms. We are not concerned with certain other gravitational theories set in flat space-time, notably those of NORDSTRÖM [1913], [1914], MIE [1915], WHITEHEAD [1922] and BIRKHOFF [1943], [1950]. For remarks on the first two of these, see PAULI [1958, p. 144].

² The reader will find a critical survey in McVITTIE [1956].

untenable in physics, as the many successes of the special theory of relativity make clear, and if Einstein's theory of gravitation is actually at fault, then what we need is a modification of that theory, not a return to Newton. If modification be needed, the first desideratum is a clear and critical understanding of what precisely are the predictions of general relativity.

We speak here of the solar field, but the mathematics applies to any spherically symmetric field in vacuo. We take the metric form to be, as in (145),

$$\Phi = \frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\sigma^2 - \left(1 - \frac{2m}{r}\right) dt^2, \quad (184)$$

$$d\sigma^2 = d\theta^2 + \sin^2 \theta d\phi^2.$$

The coordinates are very nearly, but not quite, the curvature coordinates of § 2. For simplicity of notation we have dropped the prime from the t' of (145), so that the t of (184) is not proper time at the sun's centre but for an observer in a fixed position at $r = \infty$. The coordinate r is not, of course, 'spatial distance' — it is, as earlier, defined by the statement that r^{-2} is the intrinsic Gaussian curvature of the 2-space $r = \text{const.}$, $t = \text{const.}$ As for m , it is the mass of the sun as given by (143) or, if the sun is regarded as a homogeneous fluid, by (176); but for present purposes, it is best to regard it merely as some constant.

We are about to study the orbits of planets and photons on the basis of the geodesic hypothesis. The agonist (cf. IV-§ 6) needs no encouragement to work out, as a mathematical problem, the geodesics of space-time with the metric (184). The realist, on the other hand, may have doubts. Though convinced of the validity of the geodesic hypothesis for very small bodies, he may wonder just what 'very small' means — are the Earth and Jupiter very small ¹? This question cannot be answered until a rational theory of the 2-body problem has been developed, and the only thing to do is to go ahead with the geodesic hypothesis and see what it does predict with regard to planetary motion and light rays.

The neatest way of treating geodesics is to use Lagrangian equations. If, for a general metric, we write as Lagrangian function

$$F(x, x') = \frac{1}{2} g_{ij} x'^i x'^j, \quad (185)$$

¹ Cf. remarks on smallness in II-§ 3.

the prime denoting d/dw , where w is a special parameter, the equations of geodesics read

$$\frac{d}{dw} \frac{\partial F}{\partial x^{i'}} - \frac{\partial F}{\partial x^i} = 0. \quad (186)$$

For null geodesics these equations possess the first integral

$$F = 0, \quad (187)$$

and for timelike geodesics (with $w = s$) the first integral

$$2F = -1. \quad (188)$$

The Lagrangian for (184) is given by

$$2F = \frac{r'^2}{1 - \frac{2m}{r}} + r^2(\theta'^2 + \sin^2 \theta \phi'^2) - \left(1 - \frac{2m}{r}\right) t'^2. \quad (189)$$

Since ϕ and t are ignorable coordinates, there are two first integrals,

$$\begin{aligned} \frac{\partial F}{\partial \phi'} &= r^2 \sin^2 \theta \phi' = \alpha^{-1}, \\ \frac{\partial F}{\partial t'} &= -\left(1 - \frac{2m}{r}\right) t' = -\beta, \end{aligned} \quad (190)$$

where α and β are constants depending on the initial conditions. The θ -equation reads

$$\frac{d}{dw} \frac{\partial F}{\partial \theta'} - \frac{\partial F}{\partial \theta} = \frac{d}{dw} (r^2 \theta') - r^2 \sin \theta \cos \theta \phi'^2 = 0. \quad (191)$$

We have also the first integral (187) or (188), according to the case.

It is clear from (191) that if we have initially

$$\theta = \frac{1}{2}\pi, \quad \theta' = 0, \quad (192)$$

then these equations remain true. But for any particular geodesic we can rotate the axes of reference (cf. Fig. 5, p. 266) so that (192) hold, and so there is no loss of generality, if we are discussing a single geodesic or indeed a set of 'coplanar' geodesics, in accepting (192). Then (190)

and the other first integral read

$$\begin{aligned} r^2\phi' &= \alpha^{-1}, & \left(1 - \frac{2m}{r}\right)t' &= \beta, \\ \frac{r'^2}{1 - \frac{2m}{r}} + r^2\phi'^2 - \left(1 - \frac{2m}{r}\right)t'^2 &= -\eta, \end{aligned} \quad (193)$$

where $\eta = 1$ for a timelike geodesic and $\eta = 0$ for a null geodesic.

The plan is to obtain an orbital equation connecting r and ϕ , t and w having been eliminated. We have

$$dw = \alpha r^2 d\phi, \quad \left(1 - \frac{2m}{r}\right) dt = \beta dw = \alpha \beta r^2 d\phi, \quad (194)$$

and the last of (193) gives, as an equation connecting r and ϕ ,

$$dr^2 + \left[r^2 \left(1 - \frac{2m}{r}\right) - \alpha^2 \beta^2 r^4 + \eta \alpha^2 r^4 \left(1 - \frac{2m}{r}\right) \right] d\phi^2 = 0. \quad (195)$$

We now put

$$u = \frac{1}{r}, \quad (196)$$

and, on dividing (195) by $r^4 d\phi^2$, obtain

$$\left(\frac{du}{d\phi}\right)^2 = f(u), \quad (197)$$

where

$$\begin{aligned} f(u) &= \alpha^2 \beta^2 - (u^2 + \eta \alpha^2)(1 - 2mu) \\ &= 2mu^3 - u^2 + 2\eta \alpha^2 mu + \alpha^2(\beta^2 - \eta) \\ &= 2m(u - u_1)(u - u_2)(u - u_3), \end{aligned} \quad (198)$$

u_1, u_2, u_3 being the zeros of $f(u)$, with $u_1 < u_2 < u_3$ if they are all real; we have

$$u_1 + u_2 + u_3 = \frac{1}{2m}, \quad (199)$$

$$u_2 u_3 + u_3 u_1 + u_1 u_2 = \eta \alpha^2. \quad (200)$$

The whole tale of planets and photons (according to the geodesic hypothesis) is contained in the orbital equation (197), which requires for its solution only a quadrature and an inversion; for, once (197) has been solved, w and t are given as functions of ϕ by integrating (194).

An exhaustive study of these orbits was made by HAGIHARA [1931]. We shall make a slight restriction here, considering only orbits which possess perihelia (points of closest approach to the sun). At perihelion we have

$$\frac{du}{d\phi} = 0, \quad f(u) = 0, \quad (201)$$

and for the rest of the orbit u is less than its perihelion value. By (197) $f(u) \geq 0$ throughout the orbit, and by (198) $f(u)$ is positive for large positive values of u . It follows that the three zeros of $f(u)$ are all real, with u_2 and u_3 positive, u_2 corresponding to perihelion. The graph of $f(u)$ is then of one of the two types shown in Fig. 9, with $u_1 > 0$ in Fig. 9a and $u_1 < 0$ in Fig. 9b. In the former case we get an orbit of elliptic type, with u oscillating in the range $u_1 < u < u_2$ (u_1 corresponds to aphelion). In the latter case we get an orbit of hyperbolic type. There are of course other special cases: if $u_1 = 0$ we get an orbit of parabolic type, and if $u_1 = u_2$ we get a circular orbit.

The general solution of (197) in terms of Jacobian elliptic functions is obtained by putting

$$x = \frac{1}{2}\phi \sqrt{2m(u_3 - u_1)}, \quad y = \sqrt{\frac{u - u_1}{u_2 - u_1}}, \quad k = \sqrt{\frac{u_2 - u_1}{u_3 - u_1}}. \quad (202)$$

Then (197) becomes

$$\left(\frac{dy}{dx}\right)^2 = (1 - y^2)(1 - k^2 y^2), \quad (203)$$

and the general solution of this is

$$y = \operatorname{sn}(x + \delta), \quad (204)$$

where δ is an arbitrary constant. Thus *all geodesic orbits having perihelia satisfy*

$$u - u_1 = (u_2 - u_1) \operatorname{sn}^2\left(\frac{1}{2}\phi \sqrt{2m(u_3 - u_1)} + \delta\right), \quad (205)$$

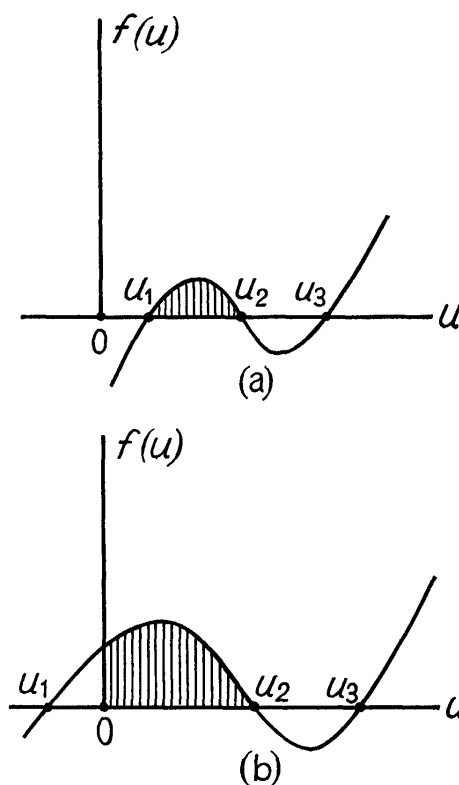


Fig 9 – Graphs of $f(u)$:

- (a) For orbit of elliptic type
(b) For orbit of hyperbolic type

where $u = 1/r$ and the modulus of the elliptic function is k as in (202).

Let us compare this result with Newtonian astronomy in the case of a planetary orbit. In making numerical estimates, we identify r with the distance from the sun's centre in the Newtonian model. Thus, using for the mass of the sun the value quoted in IV-(137) and for its radius $6.953 \times 10^{10} \text{ cm} = 2.319 \text{ sec}$, we find that at the sun's surface

$$mu = \frac{m}{r} = 2.122 \times 10^{-6}, \quad (206)$$

a very small dimensionless quantity. For more distant points, mu is still smaller — in particular at perihelion and aphelion — so that (199) gives approximately

$$2mu_3 = 1. \quad (207)$$

Thus the ratios u_1/u_3 and u_2/u_3 are small, so that, by (202), $k = 0$ approximately. But, when its modulus approaches zero, the elliptic function sn degenerates into the sine function, and so the orbital equation (205) becomes

$$u - u_1 = (u_2 - u_1)\sin^2(\tfrac{1}{2}\phi + \delta). \quad (208)$$

This is a focal conic of eccentricity $e = (u_2 - u_1)/(u_2 + u_1)$, and so an ellipse or hyperbola according as $u_1 > 0$ or $u_1 < 0$. Thus, on the basis of the geodesic hypothesis and reasonable approximations, we obtain to a high degree of approximation the outstanding fact of astronomy — the elliptical character of planetary orbits. From (194) we obtain the constancy of areal velocity.

To examine the small difference between relativistic orbit and Newtonian ellipse, we return to the exact equation (205) for the relativistic orbit. The elliptic function sn has a period $4K$, and its square sn^2 a period $2K$, where

$$K = \int_0^1 \frac{dy}{\sqrt{(1-y^2)(1-k^2y^2)}}. \quad (209)$$

Therefore the increase in ϕ between successive perihelia is accurately

$$\Delta\phi = \frac{4K}{\sqrt{2m(u_3 - u_1)}}. \quad (210)$$

We have seen above that k is small. Neglecting k^4 , (209) gives

$$K = \int_0^1 \frac{dy}{\sqrt{1-y^2}} (1 + \tfrac{1}{2}k^2y^2) = \tfrac{1}{2}\pi(1 + \tfrac{1}{4}k^2). \quad (211)$$

By (199)

$$2m(u_3 - u_1) = 1 - 2m(2u_1 + u_2), \quad (212)$$

and, since the ratios u_1/u_3 and u_2/u_3 are small, (202) gives approximately

$$k^2 = 2m(u_2 - u_1). \quad (213)$$

Thus, by (211),

$$K = \tfrac{1}{2}\pi[1 + \tfrac{1}{2}m(u_2 - u_1)], \quad (214)$$

and (210) becomes

$$\begin{aligned} \Delta\phi &= 2\pi[1 + \tfrac{1}{2}m(u_2 - u_1)][1 + m(2u_1 + u_2)] \\ &= 2\pi[1 + \tfrac{3}{2}m(u_1 + u_2)]. \end{aligned} \quad (215)$$

Since this exceeds 2π slightly, the orbit is to be regarded as an ellipse which rotates slowly in the sense in which it is described, the *advance of perihelion* per revolution of the planet around the sun being

$$\varepsilon = \Delta\phi - 2\pi = 3\pi m(u_1 + u_2) = 3\pi m\left(\frac{1}{r_1} + \frac{1}{r_2}\right), \quad (216)$$

where r_1 and r_2 correspond to the apsides (aphelion and perihelion). Since ε is very small, it is permissible to change its form in accordance with classical mechanics. Thus, if

$$\begin{aligned} a &= \text{semi-axis major of orbit,} \\ e &= \text{eccentricity of orbit,} \\ T &= \text{period (the planet's year),} \end{aligned}$$

we have

$$r_1 = a(1 + e), \quad r_2 = a(1 - e), \quad m = \frac{4\pi^2 a^3}{T^2}, \quad (217)$$

and (216) becomes

$$\varepsilon = \frac{24\pi^3 a^2}{T^2(1 - e^2)}. \quad (218)$$

Here, as throughout the book, a and T are measured in the same units; otherwise we should substitute cT for T , c being the speed of light.

EINSTEIN'S [1915c] formula (218) for the advance of perihelion per

revolution is one of the most famous formulae of general relativity ¹. As stated earlier, it is not within the scope of this book to make a pronouncement on its physical truth. The great difficulty is that the sun has not one planet, but many, and their mutual attractions are significant. The planet Mercury is the most suitable for testing purposes ²; for it, the formula (218) gives an advance of $43''.03 \pm 0.03$ per terrestrial century ³. Newtonian perturbation theory and observation both give an advance more than a hundred times that, but if the relativistic advance (218) is added to the calculated advance, there is good agreement with observation ⁴.

This mixture of the theories of Newton and Einstein is intellectually

¹ The same formula is obtained also in the theories of WHITEHEAD [1922] and BIRKHOFF [1943], [1950]. Such is its prestige that no new gravitational theory is likely to prove acceptable if it does not yield this formula, or one practically indistinguishable from it. The traditional form (218) for ε rather obscures its true connection with the metric (184). It is clear from (199) and (216) that ε depends (in our approximation) only on the value of u_3 , which is a root of the cubic equation $f(u) = 0$, with $f(u)$ as in (198). If we write $2mu_3 = x$, $\eta = 1$, this equation reads

$$x^3 - x^2 + 4m^2\alpha^2x + 4m^2\alpha^2(\beta^2 - 1) = 0.$$

From (207) we know that $x = 1 - \xi$, where ξ is small; we find at once $\xi = 4m^2\alpha^2\beta^2$, and (216) gives

$$\varepsilon = 3\pi m(u_1 + u_2) = 3\pi m\left(\frac{1}{2m} - u_3\right) = \frac{3}{2}\pi(1 - x) = \frac{3}{2}\pi\xi = 6\pi\frac{m^2}{h^2},$$

where $h = (\alpha\beta)^{-1} = r^2 d\phi/dt$, approximately. Thus *advance of perihelion is to be described as an m^2 -effect, rather than an m -effect*. This explains a well known fact that if we approximate the Schwarzschild form (184) with neglect of m^2 , we get the wrong advance of perihelion. I owe this elucidation to Mr. A. Das.

² GILVARRY [1953], [1959] has suggested the use of the asteroid Icarus, discovered in 1949, or an artificial satellite.

³ CLEMENCE [1947]; McVITTIE [1956] has $43''.15$.

⁴ Calculated advance = $5557''.18 \pm 0.85$; observed advance = $5599''.74 \pm 0.41$; difference = $42''.56 \pm 0.94$; cf. CLEMENCE [1947], McVITTIE [1956].

The complexity of the matter, and the difficulty of expressing everything in relativistic terms, is brought out by this quotation from CLEMENCE [1947]: 'The observations cannot be made in a Newtonian frame of reference. They are referred to the moving equinox, that is, they are affected by the precession of the equinoxes, and the determination of the precessional motion is one of the most difficult problems of positional astronomy, if not the most difficult. In the light of all these hazards it is not surprising that a difference of opinion could exist regarding the closeness of agreement between the observed and theoretical motions.'

repellent, since the two theories are based on such different fundamental concepts. The situation will be made clear only when the many-body problem has been handled relativistically in a rational and mathematically satisfactory way. Moreover it must be explained relative to what the perihelion is rotating. In the above theory the rotation is relative to a triad of axes which undergo parallel transport along the world-line of the sun's centre, but that is something which demands clarification from the observational standpoint.

We turn to the study of light rays, understanding by *light ray* the (r, ϕ) orbit of a null geodesic. Now $\eta = 0$ in (200) and elimination of u_3 from (199) and (200) gives

$$4mu_1 = 1 - 2mu_2 \pm [(1 + 2mu_2)^2 - 16m^2u_2^2]^{\frac{1}{2}}. \quad (219)$$

Since u_2 and u_3 are positive, (200) tells us that u_1 is negative, and so, taking the minus sign in (219), we get approximately

$$2mu_1 = -2mu_2 + 4m^2u_2^2, \quad (220)$$

and then (200) gives

$$2mu_3 = 1 - 4m^2u_2^2. \quad (221)$$

If a light ray comes in from $r = \infty$ ($u = 0$), passes the sun, and goes out again to infinity, by (197) the total increment in azimuth is

$$\Delta\phi = 2 \int_0^{u_2} \frac{du}{\sqrt{f(u)}} = 2 \int_0^{z_2} \frac{dz}{\sqrt{(z - z_1)(z - z_2)(z - z_3)}}, \quad (222)$$

where

$$z_1 = 2mu_1, \quad z_2 = 2mu_2, \quad z_3 = 2mu_3. \quad (223)$$

Inserting the approximations (220), (221), expanding, and integrating, we get, to the first order in z_2 ,

$$\begin{aligned} \Delta\phi &= 2 \int_0^{z_2} \frac{dz}{\sqrt{(z_2 - z)(z + z_2 - z_2^2)(1 - z - z_2^2)}} \\ &= 2 \int_0^{z_2} \frac{dz}{\sqrt{z_2^2 - z^2}} \left(1 + \frac{1}{2} \frac{z_2^2}{z + z_2} + \frac{1}{2} z \right) \\ &= 2(\tfrac{1}{2}\pi + \tfrac{1}{2}z_2 + \tfrac{1}{2}z_2) = \pi + 4mu_2. \end{aligned} \quad (224)$$

Here we have EINSTEIN'S [1916a] formula for the *deflection of a light ray*: it bends towards the sun through an angle

$$\Delta\phi - \pi = \frac{4m}{r_2}, \quad (225)$$

where r_2 is the minimum value of r as the ray passes the sun. With independent units of mass, length and time, this formula becomes

$$\Delta\phi - \pi = \frac{4\gamma m}{c^2 r_2}, \quad (226)$$

where γ is the gravitational constant and c the speed of light.

For light grazing the sun, (225) gives [cf. (206)] a deflection of

$$8.488 \times 10^{-6} \text{ radians} = 1''.75. \quad (227)$$

The effect of such a deflection would be to deform the apparent pattern of a group of stars when the sun passes in front of them, the stars appearing to be pushed out through a small angle inversely proportional to the angular distance from the sun's centre. The brightness of the sun prevents the observation of this effect, and it can be tested only at total eclipses of the sun. There seems to be no doubt that such an apparent deformation of the star-pattern is observed, agreeing fairly closely with (225). The reader is referred to McVITTIE [1956, p. 93] for the results of eclipse observation from 1919 to 1952, and a discussion of them.

There are three critical phenomena in the solar field:

- (i) advance of perihelion,
- (ii) deflection of light ray,
- (iii) spectral red-shift.

We have discussed the first two above; the third will be treated in the next section.

§ 9. SPECTRAL SHIFTS AND THE WORLD-FUNCTION

In III-§ 7 the relativistic theory of spectral shifts was given, and it was shown that these should, in all cases, be regarded as Doppler effects, due to relative motion of source and observer. That is, however, only a manner of speaking, and in some cases a spectral shift may usefully be split into a part due to relative motion and a part due to gravitation.

As shown in III-§ 7, there are two formulae for spectral shifts, which are of course equivalent mathematically but very different in form. Let us recall them. Fig. 10 shows the world-line C' of a source and the world-line C of an observer, with adjacent null geodesics $P'P$, $Q'Q$ joining them. Then, if $P'Q' = ds'$, $PQ = ds$, the shift is given by III-(49) as

$$\frac{\nu' - \nu}{\nu'} = 1 - \frac{ds'}{ds}, \quad (228)$$

a shift towards the red (red-shift) being positive. On the other hand, by consideration of the energy of a photon as in III-(37), we have

$$\frac{\nu' - \nu}{\nu'} = \frac{p_{i'} V^{i'} - p_i V^i}{p_{i'} V^{i'}}, \quad (229)$$

where $p^{i'}$, p^i are its 4-momenta, and $V^{i'}$, V^i the 4-velocities of source and observer, at P' and P respectively. By II-(17) and the hypothesis (already used in (229)) that p^i is tangent to $P'P$ and undergoes parallel transport along it, we can express (229) in terms of the world-function $\Omega(P'P)$ as in III-(53),

$$\frac{\nu' - \nu}{\nu'} = \frac{\Omega_{i'} V^{i'} + \Omega_i V^i}{\Omega_{i'} V^{i'}}. \quad (230)$$

The formulae (228) and (230) are the formulae for spectral shift which we shall use in what follows.

The case where source and observer are both at rest in a stationary universe is very easily dealt with by means of (228). In a stationary¹ universe the metric tensor g_{ij} does not involve the time-coordinate (x^4 or t), and space-time admits a group of motions along the t -lines. The null cone with vertex Q' is obtained from the null cone with vertex P' by simply pushing all its events up through the same increment in t . When we say that source and observer are 'at rest', we mean here that C' and C are t -lines, and so, if dt' refers to $P'Q'$ and dt to PQ , we have

$$dt' = dt. \quad (231)$$

¹ For geometrical optics in a statical universe filled with a transparent medium, see XI-§ 4.

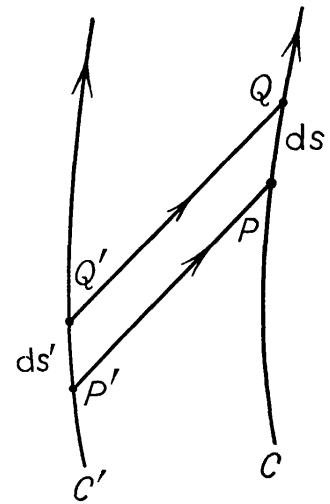


Fig. 10 – Diagram for spectral shifts

This is the key-formula. We have

$$ds'^2 = -g'_{44}dt'^2, \quad ds^2 = -g_{44}dt^2, \quad (232)$$

and so the spectral shift is

$$\frac{\nu' - \nu}{\nu'} = 1 - \sqrt{\frac{g'_{44}}{g_{44}}}. \quad (233)$$

Let us apply this to the solar field, for which

$$\Phi = \frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\sigma^2 - \left(1 - \frac{2m}{r}\right) dt^2. \quad (234)$$

With r' at source and r at observer (both fixed), (233) gives accurately

$$\frac{\nu' - \nu}{\nu'} = 1 - \sqrt{\frac{1 - \frac{2m}{r'}}{1 - \frac{2m}{r}}}. \quad (235)$$

Since m/r is small for all points outside the sun, we replace this by the approximation

$$\frac{\nu' - \nu}{\nu'} = \frac{m}{r'} - \frac{m}{r}. \quad (236)$$

If the source is an atom on the sun's surface, and the observer is on the earth (supposed fixed), we write $r' = a$ (sun's radius) and neglect the last term in (236). Thus we get EINSTEIN'S [1916a] *red-shift*,

$$\frac{\nu' - \nu}{\nu'} = \frac{m}{a}, \quad (237)$$

or, if we prefer to use independent units of mass, length, and time,

$$\frac{\nu' - \nu}{\nu'} = \frac{\gamma m}{c^2 a}. \quad (238)$$

This dimensionless quantity is precisely one quarter of the deflection (225) for light grazing the sun, and its numerical value is 2.122×10^{-6} . Spectral shifts are sometimes expressed in km sec^{-1} , and since

$$1 = 2.998 \times 10^5 \text{km sec}^{-1}, \quad (239)$$

the red-shift (237) due to the sun's field is $0.636 \text{ km sec}^{-1}$.

McVITTIE [1956, p. 97] gives a table of red-shifts observed in the solar spectrum. There is some disagreement between theory and

observation, the theoretical value (237) being approached only near the limb of the sun, and the observed shift decreasing towards the centre of the disc; according to the theory given above, the position of the source on the sun's surface should make no difference at all. For the very dense companion of Sirius, the red-shift should be about thirty times as great as for the sun; it has been widely stated that there is general agreement between observation and theory in that case ¹.

The simple formula (233) applies only to a stationary universe with source and observer fixed in it. Allowance for the motions of source and observer may be made crudely by adding the Doppler effect due to relative radial motion in flat space-time, but for a satisfactory treatment of red-shift it is better to start all over again without the stationary hypothesis, using (230) instead of (228). If only the world-function Ω were known, (230) would give the spectral shift accurately for any velocities of source and observer.

We shall proceed on the assumption that the gravitational field is weak (space-time of small curvature), and start by calculating the world-function correct to the first order. This has nothing to do with the velocities of source and observer, and is of interest apart from problems of spectral shift. At a later stage we shall make a further approximation based on smallness of velocities.

Let us use coordinates x^i for which

$$\begin{aligned} g_{ij} &= \eta_{ij} + \gamma_{ij}, \\ \eta_{ij} &= \text{diag}(1, 1, 1, -1), \end{aligned} \quad (240)$$

the γ 's being small (O_1). To fix the ideas, we shall regard x^i as rectangular Cartesians in Euclidean 4-space (Fig. 11), so that we must distinguish between a *geodesic* with equations

$$\frac{d^2 x^i}{dw^2} + \Gamma_{jk}^i \frac{dx^j}{dw} \frac{dx^k}{dw} = 0, \quad (241)$$

and a *straight line* with linear

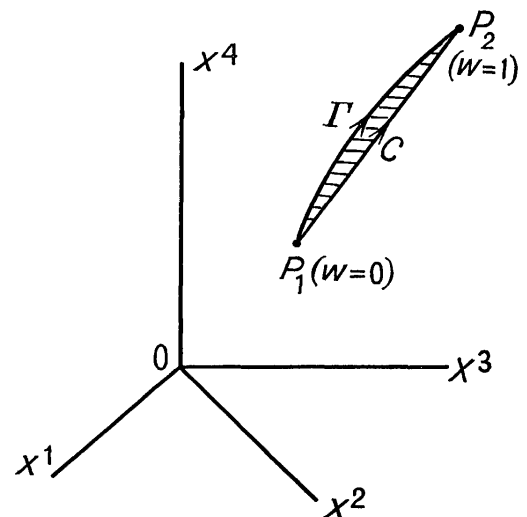


Fig. 11 — A geodesic Γ and an adjacent straight line C

¹ TOLMAN [1934b, p. 212], BERGMANN [1942, p. 222], MØLLER [1952, p. 348]. But McVITTIE [1956, p. 98] finds the theoretical value more than twice too large. See also FINLAY-FREUNDLICH [1953] [1954a, b].

equations. Let P_1 and P_2 be two events, joined by a geodesic Γ (perhaps null) and a straight line C . For Γ we have the equations (241), with w a special parameter, running from 0 at P_1 to 1 at P_2 , and for C

$$x^i = (1 - w)x^{i_1} + wx^{i_2}, \quad (242)$$

with w again running from 0 to 1; x^{i_1} and x^{i_2} are the coordinates of P_1 and P_2 respectively. We set up a correspondence between the events on Γ and C by associating events with the same values of w .

On account of the near-flatness, Γ lies close to C , and in fact C may be regarded as a variation of Γ . From the definition of a geodesic as in I-§ 2 it follows that

$$\int_{\Gamma} g_{ij} \frac{dx^i}{dw} \frac{dx^j}{dw} dw = \int_C g_{ij} \frac{dx^i}{dw} \frac{dx^j}{dw} dw + O_2. \quad (243)$$

Then by II-(1) the world-function is

$$\begin{aligned} \Omega(P_1 P_2) &= \frac{1}{2} \int_{\Gamma} g_{ij} \frac{dx^i}{dw} \frac{dx^j}{dw} dw \\ &= \frac{1}{2} \int_C g_{ij} \frac{dx^i}{dw} \frac{dx^j}{dw} dw + O_2 \\ &= \frac{1}{2} \Delta x^i \Delta x^j \int_C g_{ij} dw + O_2, \end{aligned} \quad (244)$$

where

$$\Delta x^i = x^{i_2} - x^{i_1}. \quad (245)$$

We have in fact

$$\Omega(P_1 P_2) = \frac{1}{2} \eta_{ij} \Delta x^i \Delta x^j + \frac{1}{2} \Delta x^i \Delta x^j \int_C \gamma_{ij} dw + O_2, \quad (246)$$

the first part being the world-function for flat space-time and the second part being O_1 .

We assume that γ_{ij} are assigned functions of the x 's. In evaluating the integral in (246) we are to substitute from (242), so that we may write

$$\gamma_{ij}(x) = f_{ij}(P_1, P_2, w), \quad (246)$$

and the world-function reads (we drop the second-order error)

$$\Omega(P_1 P_2) = \frac{1}{2} \eta_{ij} \Delta x^i \Delta x^j + \frac{1}{2} \Delta x^i \Delta x^j \int_0^1 f_{ij}(P_1, P_2, w) dw. \quad (247)$$

All we have to do is to evaluate the integral along the straight line C .

To calculate a spectral shift we need, not the world-function itself, but its partial derivatives. To differentiate (247), we note that

$$(\Delta x^i)_{,j_1} = -\delta_{ij}, \quad (\Delta x^i)_{,j_2} = \delta_{ij}, \quad (248)$$

and, by (242) and (246), for fixed value of w ,

$$f_{ij,k_1} = (1-w)\gamma_{ij,k}, \quad f_{ij,k_2} = w\gamma_{ij,k}. \quad (249)$$

Hence

$$\begin{aligned} \Omega_{k_1} &= -\eta_{jk}\Delta x^j - \Delta x^j \int_0^1 \gamma_{jk} dw + \frac{1}{2} \Delta x^i \Delta x^j \int_0^1 \gamma_{ij,k} (1-w) dw, \\ \Omega_{k_2} &= \eta_{jk}\Delta x^j + \Delta x^j \int_0^1 \gamma_{jk} dw + \frac{1}{2} \Delta x^i \Delta x^j \int_0^1 \gamma_{ij,k} w dw. \end{aligned} \quad (250)$$

Although we do not need them for spectral shifts, we might as well proceed to get the covariant second-order derivatives of Ω , which (following the notation of Chap. II) will be denoted by subscripts without strokes. We have

$$\begin{aligned} \Omega_{k_1 m_1} &= \Omega_{k_1, m_1} - \Gamma_{k_1 m_1}^{a_1} \Omega_{a_1}, \\ \Omega_{k_1 m_2} &= \Omega_{m_2 k_1} = \Omega_{k_1, m_2} = \Omega_{m_2, k_1}, \\ \Omega_{k_2 m_2} &= \Omega_{k_2, m_2} - \Gamma_{k_2 m_2}^{a_2} \Omega_{a_2}, \end{aligned} \quad (251)$$

where the secondary (numerical) suffixes on the Γ 's indicate evaluation at P_1 or P_2 , as the case may be, and so

$$\begin{aligned} \Omega_{k_1 m_1} &= \eta_{km} + \int_0^1 \gamma_{km} dw - \Delta x^j \int_0^1 (\gamma_{jk,m} + \gamma_{jm,k})(1-w) dw \\ &\quad + \frac{1}{2} \Delta x^i \Delta x^j \int_0^1 \gamma_{ij,km} (1-w)^2 dw + \Gamma_{k_1 m_1}^{a_1} \eta_{aj} \Delta x^j, \\ \Omega_{k_1 m_2} &= \Omega_{m_2 k_1} = -\eta_{km} - \int_0^1 \gamma_{km} dw - \Delta x^j \int_0^1 \gamma_{jk,m} w dw \\ &\quad + \Delta x^j \int_0^1 \gamma_{jm,k} (1-w) dw + \frac{1}{2} \Delta x^i \Delta x^j \int_0^1 \gamma_{ij,km} w (1-w) dw, \\ \Omega_{k_2 m_2} &= \eta_{km} + \int_0^1 \gamma_{km} dw + \Delta x^j \int_0^1 (\gamma_{jk,m} + \gamma_{jm,k}) w dw \\ &\quad + \frac{1}{2} \Delta x^i \Delta x^j \int_0^1 \gamma_{ij,km} w^2 dw - \Gamma_{k_2 m_2}^{a_2} \eta_{aj} \Delta x^j. \end{aligned} \quad (252)$$

In these formulae there is an O_2 error. We may compare (252) with II-(95), which were based on the same approximation (small curvature), but in which the coordinate system was general.

For a weak static field, the γ 's are independent of x^4 (or t) and $\gamma_{\alpha 4} = 0$. This makes no great simplification if the expression (246) for Ω , but it is worth while to write out the partial derivatives (250):

$$\begin{aligned}
 \Omega_{\gamma_1} &= -\Delta x^\gamma - \Delta x^\beta \int_0^1 \gamma_{\beta\gamma} dw + \frac{1}{2} \Delta x^\alpha \Delta x^\beta \int_0^1 \gamma_{\alpha\beta,\gamma} (1-w) dw \\
 &\quad + \frac{1}{2} (\Delta t)^2 \int_0^1 \gamma_{44,\gamma} (1-w) dw, \\
 \Omega_{4_1} &= \Delta t (1 - \int_0^1 \gamma_{44} dw) = -\Delta t \int_0^1 g_{44} dw, \\
 \Omega_{\gamma_2} &= \Delta x^\gamma + \Delta x^\beta \int_0^1 \gamma_{\beta\gamma} dw + \frac{1}{2} \Delta x^\alpha \Delta x^\beta \int_0^1 \gamma_{\alpha\beta,\gamma} w dw \\
 &\quad + \frac{1}{2} (\Delta t)^2 \int_0^1 \gamma_{44,\gamma} w dw, \\
 \Omega_{4_1} &= -\Omega_{4_2}.
 \end{aligned} \tag{253}$$

The last equation tells us that the t -derivatives of Ω at P_1 and P_2 differ only in sign; on account of the existence of a group of motions this holds not only in the static case but also in the stationary case, and we might have used this fact to deduce (233) from (230).

For the solar field the metric (234) may be written

$$\Phi = dx^\alpha dx^\alpha + \frac{\frac{2m}{r^3}}{1 - \frac{2m}{r}} (x^\alpha dx^\alpha)^2 - \left(1 - \frac{2m}{r}\right) dt^2, \quad r^2 = x^\alpha x^\alpha, \tag{254}$$

so that, dropping an O_2 term, we have

$$\begin{aligned}
 g_{ij} &= \eta_{ij} + \gamma_{ij}, \\
 \gamma_{\alpha\beta} &= \frac{2m x^\alpha x^\beta}{r^3} = 2m(r^{-1} \delta_{\alpha\beta} - r_{,\alpha\beta}), \\
 \gamma_{\alpha 4} &= 0, \quad \gamma_{44} = \frac{2m}{r}.
 \end{aligned} \tag{255}$$

By (246) the world-function for the solar field is

$$\begin{aligned} \Omega(P_1 P_2) = & \frac{1}{2}(\Delta x^\alpha \Delta x^\alpha - (\Delta t)^2) \\ & + m \Delta x^\alpha \Delta x^\beta \int_0^1 \frac{x^\alpha x^\beta}{r^3} dw + m(\Delta t)^2 \int_0^1 \frac{dw}{r} + O_2. \end{aligned} \quad (256)$$

The values of these integrals are given on p. 308. The first derivatives of Ω are given by (253), in which we are to substitute from (255). In making calculations it is well to have before one a diagram (Fig. 12) showing a Euclidean 3-space in which x^α are rectangular Cartesians.

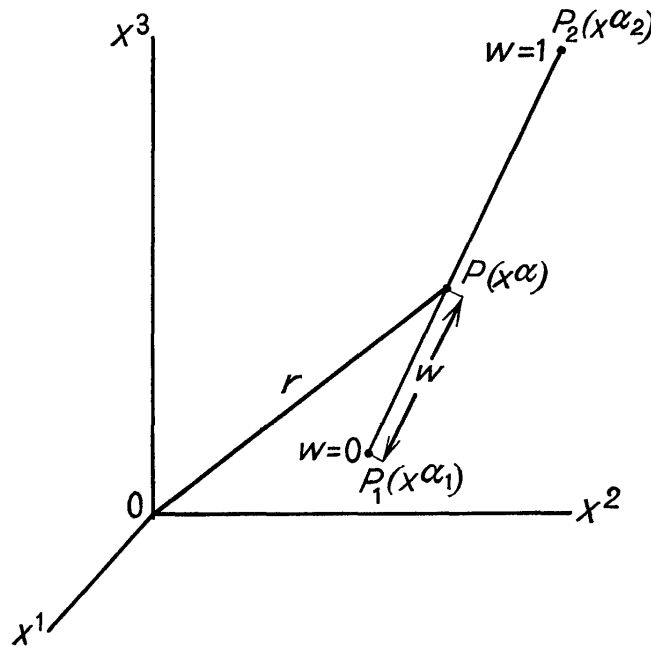


Fig. 12 – Space-diagram for use with world-function

So much for the calculation of Ω and its derivatives in a weak field, in particular a static one, and more particularly the solar field. When the derivatives have been found, the spectral shift is given by (230). But a complication arises. In any particular problem, the event of emission P_1 and the event of reception P_2 (shown as P' , P in Fig. 10) cannot both be chosen arbitrarily, since $P_1 P_2$ is a null geodesic. We start by specifying P_2 , i.e. assigning the coordinates x^{i_2} . With vertex P_2 , we draw the null cone into the past, cutting the world-line of the source at P_1 . It is clear that the independent quantities are

$$x^{i_2}, \quad \Delta x^\alpha, \quad (257)$$

and that $\Delta x^4 (= \Delta t)$ is determined by them in a given space-time. To find it, we remark that $\Omega(P_1 P_2) = 0$ since $P_1 P_2$ is a null geodesic,

and so the basic partial differential equation II-(20) gives

$$g^{k_2 m_2} \Omega_{k_2} \Omega_{m_2} = 0. \quad (258)$$

Since (240) gives, to the first order,

$$g^{ij} = \eta_{ij} - \eta_{ia} \gamma_{ab} \eta_{bj}, \quad (259)$$

(258) may be written, by (250),

$$\eta_{km} \Omega_{k_2} \Omega_{m_2} - \gamma_{k_2 m_2} \Delta x^k \Delta x^m = 0. \quad (260)$$

Using (250) again, we get

$$\eta_{km} \Delta x^k \Delta x^m = Q, \quad (261)$$

where

$$Q = \gamma_{k_2 m_2} \Delta x^k \Delta x^m - 2 \Delta x^j \Delta x^k \int_0^1 \gamma_{jk} dw - \Delta x^i \Delta x^j \Delta x^k \int_0^1 \gamma_{ij,k} w dw. \quad (262)$$

Hence

$$\begin{aligned} (\Delta t)^2 &= \Delta x^\alpha \Delta x^\alpha - Q, \\ \Delta t &= (\Delta x^\alpha \Delta x^\alpha)^{\frac{1}{2}} - \frac{1}{2} Q (\Delta x^\alpha \Delta x^\alpha)^{-\frac{1}{2}}. \end{aligned} \quad (263)$$

The last term here is small, and in Q we may substitute

$$\Delta t = (\Delta x^\alpha \Delta x^\alpha)^{\frac{1}{2}}. \quad (264)$$

We omit throughout O_2 error terms.

All this is complicated, but one feels justified in pursuing the detail in view of the astronomical importance of spectral shifts. To clarify the procedure, let us sum it up, and then apply it to the case of small velocities of source and observer.

The steps are as follows:

- (i) Assign the field, i.e. the small functions $\gamma_{ij}(x)$.
- (ii) Choose an event of reception $P_2(x^{i_2})$.
- (iii) Assign Δx^α , thus fixing the position of the emission, but not its time.
- (iv) Calculate ${}^1 \Delta t$ from (263), thus getting the event P_1 of emission.
- (v) Calculate ${}^1 \Omega_{k_1}$ and Ω_{k_2} from (250) as functions of the seven quantities (257).

¹ In these calculations, the integrals are evaluated on the straight line of Fig. 12, with due allowance for the linear change in t in the case of non-stationary fields.

(vi) Assign two unit 4-vectors to represent the 4-velocities V^{i_1} , V^{i_2} of source and observer.

(vii) Calculate the spectral shift from (230).

Suppose that V^{α_1} and V^{α_2} are small (we neglect products)¹. Then for either

$$g_{44}(V^4)^2 = -1, \quad (265)$$

so that

$$V^4 = (1 - \gamma_{44})^{-\frac{1}{2}} = 1 + \frac{1}{2}\gamma_{44}. \quad (266)$$

Accordingly

$$\Omega_k V^k = \Omega_\alpha V^\alpha + \Omega_4(1 + \frac{1}{2}\gamma_{44}), \quad (267)$$

and so, by (250),

$$\begin{aligned} \Omega_{k_1} V^{k_1} &= \Omega_{4_1} + \frac{1}{2}\gamma_{4_1 4_1} \Delta t - V^{\alpha_1} \Delta x^\alpha, \\ \Omega_{k_2} V^{k_2} &= \Omega_{4_2} - \frac{1}{2}\gamma_{4_2 4_2} \Delta t + V^{\alpha_2} \Delta x^\alpha, \end{aligned} \quad (268)$$

the first terms on the right being finite and the others small. Adding, and using (250),

$$\begin{aligned} \Omega_{k_1} V^{k_1} + \Omega_{k_2} V^{k_2} &= (V^{\alpha_2} - V^{\alpha_1}) \Delta x^\alpha + \frac{1}{2}(\gamma_{4_1 4_1} - \gamma_{4_2 4_2}) \Delta t \\ &\quad + \frac{1}{2} \Delta x^i \Delta x^j \int_0^1 \gamma_{ij,4} dw, \quad (269) \\ \Omega_{k_1} V^{k_1} &= \Delta t + O_1. \end{aligned}$$

Then (230) gives for the small spectral shift (red-shift positive)

$$\begin{aligned} \frac{\nu_1 - \nu_2}{\nu_1} &= (V^{\alpha_2} - V^{\alpha_1}) \frac{\Delta x^\alpha}{\Delta t} + \frac{1}{2}(\gamma_{4_1 4_1} - \gamma_{4_2 4_2}) \\ &\quad + \frac{1}{2} \frac{\Delta x^\alpha \Delta x^\beta}{\Delta t} \int_0^1 \gamma_{\alpha\beta,4} dw + \Delta x^\alpha \int_0^1 \gamma_{\alpha 4,4} dw + \frac{1}{2} \Delta t \int_0^1 \gamma_{44,4} dw, \quad (270) \end{aligned}$$

¹ In fact, we treat the velocity components as small of the first order (O_1), like the γ 's, and omit O_2 terms. However, in the solar system, the speeds of the planets are of order $(m/r)^{\frac{1}{2}}$ while the γ 's are of order m/r (m = mass of sun, r = distance from its centre; $m/r = 2 \times 10^{-6}$ at sun's surface). This suggests an approximation in which V^{α_1} and V^{α_2} are $O_{\frac{1}{2}}$ and the γ 's are O_1 . Such an approximation can be worked out without much additional labour. The result is to augment the right hand side of (270) with the following term:

$$\frac{\Delta x^\alpha \Delta x^\beta}{\Delta t^2} (V^{\alpha_2} - V^{\alpha_1}) V^{\beta_1} + \frac{1}{2}(V^{\alpha_1} V^{\alpha_1} - V^{\alpha_2} V^{\alpha_2}). \quad (270a)$$

where Δt is as in (264), the approximate 'distance' between source and observer.

In (270) the first term on the right represents a Doppler effect due to relative radial motion, the second is a gravitational effect (as in the case of the solar spectrum), while the other terms are due to change of the field with time. These last terms disappear if we add a further assumption to the effect that the field changes very slowly with time. However, some care is needed here, on account of the factors which become large if Δt is large. In our approximation we have simply regarded Δt as a finite quantity, but it will be large in the case of a distant star, and then we might not feel justified in dropping the last terms in (270). But indeed the whole question of approximations for weak fields in large domains is too complicated to discuss here¹. The present chapter is devoted to space-times with spherical symmetry, and we have allowed ourselves to wander away from the solar field only in order to display the problem of spectral shifts in that field against a more general background.

Returning to the formula (256) for the solar field, we note that

$$\Delta x^\alpha \Delta x^\beta \int_0^1 \frac{x^\alpha x^\beta}{r^3} dw = P_1 P_2 \left(\log \frac{\tan \frac{1}{2}\theta_1}{\tan \frac{1}{2}\theta_2} + \cos \theta_1 - \cos \theta_2 \right), \quad (271)$$

$$\int_0^1 \frac{dw}{r} = P_1 P_2 \log \frac{\tan \frac{1}{2}\theta_1}{\tan \frac{1}{2}\theta_2},$$

where $P_1 P_2$ is the Euclidean distance in Fig. 12 and θ_1, θ_2 are the angles which OP_1, OP_2 make with $P_1 P_2$.

¹ See XI-§§ 5, 6 for astronomical observations and stellar aberration. In XI-§ 9 spectral shift is discussed with an approximation based on the closeness of source and observer, and not on smallness of curvature of space-time.

CHAPTER VIII

SOME SPECIAL UNIVERSES

§ 1. AXIAL SYMMETRY

In Newtonian physics the axial symmetry of a gravitational field is easy to define: if we use cylindrical coordinates (r, ϕ, z) with $r = 0$ on the axis of symmetry, then the gravitational potential is independent of the azimuthal angle ϕ .

In attempting to carry this idea over into relativity, we are led to consider a universe in which the metric tensor g_{ij} is independent of one of the coordinates, that coordinate (ϕ) being cyclic in the sense that we regain the same event if we increase it by 2π , the other three coordinates being held fixed. In fact, space-time admits a group of motions along the ϕ -lines.

It is impossible to obtain any results of interest in such a general situation, in which we have ten functions of three coordinates. If we introduce a stationary condition, we make g_{ij} independent of both ϕ and t , but that is not enough, for we still have ten functions to deal with.

We therefore go a step further and suppose ϕ and t to be reversible in the sense that the metric is unchanged if we replace ϕ by $-\phi$ or t by $-t$. In physical terms, this means that we are dealing with matter which is not rotating.¹ Mathematically it means that the metric form contains $d\phi$ and dt only as squares, so that the form reads

$$\begin{aligned}\Phi &= \Psi + g_{33}(dx^3)^2 + g_{44}(dx^4)^2, \\ \Psi &= g_{11}(dx^1)^2 + 2g_{12}dx^1dx^2 + g_{22}(dx^2)^2,\end{aligned}\tag{1}$$

where the g 's are functions of (x^1, x^2) ; we have written x^3 for ϕ and x^4 for t . We count now only five unknown functions. But we can at

¹ The field of a rotating body, in the linear approximation, was studied by LENSE and THIRRING [1918] and THIRRING [1918], [1921]. See also STOCKUM [1937], [1938], CLARK [1947a], [1948], [1949e], [1950a], [1950b], DAS [1957a].

once reduce them to *three*¹. In the ingenious argument² which follows, an essential step is to use coordinates (x^1, x^2) for which Ψ has the isothermal form:

$$\Psi = \alpha^2[(dx^1)^2 + (dx^2)^2], \quad (2)$$

α being a function of (x^1, x^2) . Thus, with a slight change in notation, (1) becomes

$$\Phi = \alpha^2[(dx^1)^2 + (dx^2)^2] + \beta^2(dx^3)^2 - \gamma^2(dx^4)^2, \quad (3)$$

where (α, β, γ) are functions of (x^1, x^2) .

By straightforward but tedious calculation using 1-(106), we find the surviving components of the Ricci tensor to have the following values:

$$\begin{aligned} R_{11} &= \left(\frac{\alpha_1}{\alpha}\right)_1 + \left(\frac{\alpha_2}{\alpha}\right)_2 + \frac{\beta_{11}}{\beta} + \frac{\gamma_{11}}{\gamma} \\ &\quad + \frac{\alpha_2}{\alpha} \left(\frac{\beta_2}{\beta} + \frac{\gamma_2}{\gamma}\right) - \frac{\alpha_1}{\alpha} \left(\frac{\beta_1}{\beta} + \frac{\gamma_1}{\gamma}\right), \\ R_{22} &= \left(\frac{\alpha_1}{\alpha}\right)_1 + \left(\frac{\alpha_2}{\alpha}\right)_2 + \frac{\beta_{22}}{\beta} + \frac{\gamma_{22}}{\gamma} \\ &\quad + \frac{\alpha_1}{\alpha} \left(\frac{\beta_1}{\beta} + \frac{\gamma_1}{\gamma}\right) - \frac{\alpha_2}{\alpha} \left(\frac{\beta_2}{\beta} + \frac{\gamma_2}{\gamma}\right), \quad (4) \\ R_{12} &= \frac{\beta_{12}}{\beta} + \frac{\gamma_{12}}{\gamma} - \frac{\alpha_2}{\alpha} \left(\frac{\beta_1}{\beta} + \frac{\gamma_1}{\gamma}\right) - \frac{\alpha_1}{\alpha} \left(\frac{\beta_2}{\beta} + \frac{\gamma_2}{\gamma}\right), \\ R_{33} &= \frac{\beta}{\alpha^2} \left\{ \Delta\beta + \frac{1}{\gamma} (\beta_1\gamma_1 + \beta_2\gamma_2) \right\}, \\ R_{44} &= -\frac{\gamma}{\alpha^2} \left\{ \Delta\gamma + \frac{1}{\beta} (\beta_1\gamma_1 + \beta_2\gamma_2) \right\}, \end{aligned}$$

where the subscripts on the right indicate partial derivatives with respect to x^1 and x^2 , and

$$\Delta\beta = \beta_{11} + \beta_{22}, \quad \Delta\gamma = \gamma_{11} + \gamma_{22}. \quad (5)$$

¹ BERGMANN [1942], p. 206 states that, on the basis of symmetry alone, it is possible to obtain a form with only *two* unknown functions, but that is not correct; the reduction to two involves the use of some of the field equations in vacuo.

² WEYL [1917], [1919c], LEVI-CIVITA [1917a]–[1919], BACH [1922], CHAZY [1923], [1924], DARMOIS [1927].

We note that

$$R_3^3 + R_4^4 = \beta^{-2}R_{33} - \gamma^{-2}R_{44} = \frac{1}{\alpha^2\beta\gamma}\Delta(\beta\gamma). \quad (6)$$

All this is true whether matter be present or not. We now examine a domain in which there is no matter, so that the field equations are

$$R_{ij} = 0, \quad (7)$$

and we obtain from (6) the remarkable result

$$\Delta(\beta\gamma) = 0. \quad (8)$$

This means that $\beta\gamma$ is a harmonic function of (x^1, x^2) . Write

$$\beta\gamma = r(x^1, x^2); \quad (9)$$

then there exists a conjugate harmonic function $z(x^1, x^2)$, such that

$$r + iz = f(x^1 + ix^2), \quad (10)$$

where f is an analytic function. We now make a transformation

$$(x^1, x^2) \rightarrow (r, z). \quad (11)$$

Since this transformation is conformal, it preserves the isothermal character of a quadratic form, and so

$$\alpha^2[(dx^1)^2 + (dx^2)^2] = A(dr^2 + dz^2), \quad (12)$$

where A is a function of (r, z) . Further, by (9) we have

$$\beta = r/\gamma, \quad (13)$$

and so the Φ of (3) becomes a form with only *two* arbitrary functions in it.

To state the result compactly, let us forget the old meanings of (x^1, x^2) and write

$$x^1 = r, \quad x^2 = z, \quad x^3 = \phi, \quad x^4 = t. \quad (14)$$

Let us also forget the old meanings of (α, β, γ) . Then we assert that, *in any domain in which the conditions of axial symmetry (as here understood) are satisfied, and in which*¹

$$R_3^3 + R_4^4 = 0, \quad (15)$$

¹ Although we wrote down the whole set of vacuum equations in (7), we used only the combination (15).

the metric form is reducible to

$$\Phi = \alpha^2(dr^2 + dz^2) + r^2\gamma^{-2}d\phi^2 - \gamma^2dt^2, \quad (16)$$

where (α, γ) are functions of (r, z) .

Now (16) is really the same form as (3) except for the relation (13). We can therefore use (4) to evaluate the Ricci tensor, inserting (13) to eliminate β . However, the formalism is important, and it is wise to change from the notation (α, γ) to (λ, ν) , putting

$$\alpha = e^{\nu-\lambda}, \quad \beta = re^{-\lambda}, \quad \gamma = e^\lambda, \quad (17)$$

so that the metric form reads

$$\Phi = e^{2(\nu-\lambda)}(dr^2 + dz^2) + r^2e^{-2\lambda}d\phi^2 - e^{2\lambda}dt^2. \quad (18)$$

From (4) we find

$$\begin{aligned} \frac{1}{2}(R_{11} + R_{22}) &= \Delta\nu - \left(\Delta\lambda + \frac{\lambda_1}{r}\right) + \lambda_1^2 + \lambda_2^2, \\ \frac{1}{2}(R_{11} - R_{22}) &= \lambda_1^2 - \lambda_2^2 - \frac{\nu_1}{r}, \\ R_{12} &= 2\lambda_1\lambda_2 - \frac{\nu_2}{r}, \\ R_3^3 - R_4^4 &= -\frac{2}{\alpha^2}\left(\Delta\lambda + \frac{\lambda_1}{r}\right), \\ R_3^3 + R_4^4 &= 0, \end{aligned} \quad (19)$$

the last being of course known already.

Imposing the complete set of vacuum equations (7), we get

$$\Delta\lambda + \frac{\lambda_1}{r} = 0, \quad (20)$$

$$\nu_1 = r(\lambda_1^2 - \lambda_2^2), \quad \nu_2 = 2r\lambda_1\lambda_2, \quad (21)$$

$$\Delta\nu + \lambda_1^2 + \lambda_2^2 = 0. \quad (22)$$

If (20) is satisfied, then (21) are integrable, and (22) is implied by the other equations. Thus, *in any domain E in which the vacuum equations are satisfied we obtain fields by the following prescription: For λ choose any solution of (20) and define ν by*

$$\nu = \int r[(\lambda_1^2 - \lambda_2^2)dr + 2\lambda_1\lambda_2dz], \quad (23)$$

the path of integration lying in E .

We now come to the most amazing fact in this work: written out explicitly in the form

$$\frac{\partial^2 \lambda}{\partial r^2} + \frac{1}{r} \frac{\partial \lambda}{\partial r} + \frac{\partial^2 \lambda}{\partial z^2} = 0, \quad (24)$$

the equation (20) is recognized as Laplace's equation in cylindrical coordinates (r, ϕ, z) in Euclidean 3-space for a function which is independent of ϕ . This gives an easy way of finding solutions of (20).

The physical purpose of all this work is to study the gravitational field of a non-rotating body with axial symmetry. Taking (r, ϕ, z) as

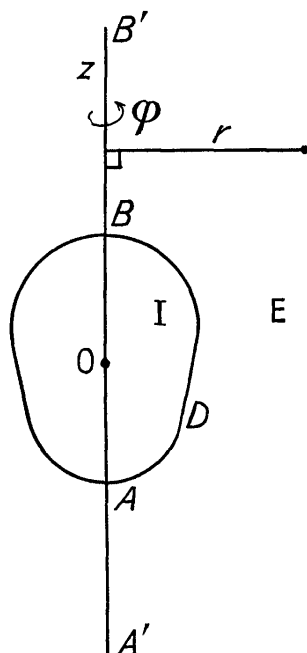


Fig. 1 – Body with
axial symmetry
viewed in Euclidean
3-space

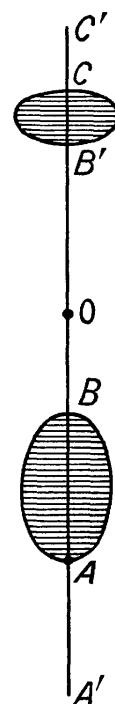


Fig. 2 –
No static
solution for
two bodies!

cylindrical coordinates in Euclidean 3-space, we sketch in Fig. 1 a body with interior domain I and exterior domain E . We expect the metric form in E to be as in (18) with $\lambda(r, z)$ some harmonic function and $\nu(r, z)$ defined by (23). The metric in I is another matter altogether.

But before thinking about I we must examine the question of *elementary flatness* in E . This demands that, for any infinitesimal spacelike circle, the ratio of circumference to radius shall be 2π . The dangerous place is the z -axis; if we take a small circle on which (r, z, t)

are constant, with r infinitesimal, it is easy to see from (18) that the requirement is

$$\nu = 0 \text{ for } r = 0. \quad (25)$$

Now (23) defines ν only to within an additive constant — let us choose $\nu = 0$ at A . Then $\nu = 0$ on AA' , but at B

$$\nu = \int_{ADB} r \left\{ \left[\left(\frac{\partial \lambda}{\partial r} \right)^2 - \left(\frac{\partial \lambda}{\partial z} \right)^2 \right] dr + 2 \frac{\partial \lambda}{\partial r} \frac{\partial \lambda}{\partial z} dz \right\}, \quad (26)$$

the integral (which is independent of path in E) being taken along the boundary ADB , or any curve deformable into it. Since ν receives no increment on BB' , the path ADB may be changed into an infinite semicircle. It is clear that the condition of elementary flatness is satisfied if λ vanishes at infinity at least as fast as $(r^2 + z^2)^{-\frac{1}{2}}$.

Let us now turn to Fig. 2, which shows two bodies. If λ behaves as stated and $\nu = 0$ at A , then $\nu = 0$ at C . But there is no reason to suppose that $\nu = 0$ on BB' , and so the proposed solution of the two-body problem fails. It is well for the theory of relativity that it does fail, for if it succeeded we would see two massive bodies permanently at rest in spite of their mutual gravitational attraction!

In Newtonian theory it is not possible for a system of free particles to be in equilibrium under their mutual attractions. But configurations of equilibrium are possible if we admit *both positive and negative masses*¹, with the inverse square law, like masses attracting and unlike masses repelling. This suggests the investigation of axially symmetric relativistic fields corresponding to 'particles' with constant 'masses' m_1, m_2, \dots (not necessarily positive) situated on the z -axis at z_1, z_2, \dots . Accordingly we take

$$\lambda = -\frac{m_1}{\rho_1} - \frac{m_2}{\rho_2} - \dots, \quad (27)$$

$$\rho_1^2 = r^2 + (z - z_1)^2, \quad \rho_2^2 = r^2 + (z - z_2)^2, \quad \dots$$

Then ν can be calculated from (26). For two particles, we get (cf. CURZON [1924a, b])

$$\nu = -\frac{m_1^2 r^2}{2\rho_1^4} - \frac{m_2^2 r^2}{2\rho_2^4} + \frac{2m_1 m_2}{(z_1 - z_2)^2} \left[\frac{r^2 + (z - z_1)(z - z_2)}{\rho_1 \rho_2} - 1 \right]. \quad (28)$$

¹ Cf. BONDI [1957b].

For an arbitrary number of particles, ν is given by a similar but more complicated formula. It is easy to see that (28) makes $\nu = 0$ for $r = 0$ provided z lies *outside* the range (z_1, z_2) , but not if it lies in that range. Consequently we have what may be called 'two particles connected by a strut'. In the general case, we may likewise speak of a strut holding the particles in position, but we may avoid this dubious expression by saying that (27) gives an axially symmetric vacuum field in a domain E which excludes the segment of the z -axis joining the two extreme particles. We are not concerned with the interior domain I in which this segment lies. But let us reduce I to a set of small spheres of radius a with centres at the particles, and work with the unattained limit in which a is very small. The domain E now includes the portions of the z -axis between these spheres, and we have a regular field in E *provided* $\nu = 0$ *on those portions*. The criterion for this is that the integral (26) should vanish when taken over each of the small semi-circles of radius a and centres at z_1, z_2, \dots . To investigate this integral for the semi-circle at z_1 , say, we write (27) in the form $\lambda = -m_1/\rho_1 + \lambda'_1$, where λ'_1 is finite for $r = 0, z = z_1$. Then the integral (26) breaks into three parts. The first part appears to be large of order a^{-2} , but actually vanishes; the second part in general is finite; and the third part vanishes with a . All we need then is the vanishing of the second part, and we find that the condition for this is

$$\frac{\partial \lambda'_1}{\partial z} = 0 \text{ for } z = z_1. \quad (29)$$

Thus $\nu = 0$ on the axis, and we have a regular field in E (i.e. elementary flatness everywhere in E) provided conditions of the type (29) are satisfied, one for each particle. The remarkable thing is that, since λ is formally the sum of the Newtonian potentials of the particles, these conditions are identical with the Newtonian conditions of equilibrium — the resultant force on each particle must vanish. We may say that, when the constants representing masses (positive and negative) and positions satisfy the stated conditions, the struts referred to above are not required for the maintenance of equilibrium¹.

However, although the above work is interesting and suggestive,

¹ Spherical symmetry is a particular case of axial symmetry. To derive the Schwarzschild form, one takes for λ the potential of a rod; cf. EREZ and ROSEN [1959], where the field of a quadripole particle is also discussed.

it has two defects. First, in a field theory we do not like to deal with concentrated particles (the unattained limit $a \rightarrow 0$), and secondly negative masses appear to be unphysical. What we would like to obtain is a set of models, each representing a body of reasonable nature surrounded by empty space. The simplest model is that in which we take for the interior domain I a sphere $r^2 + z^2 \leq a^2$ filled with matter of constant 'density' σ , and define λ as the Newtonian potential $\lambda = -\int \rho^{-1} \sigma dI$, where ρ is Euclidean distance in the (r, z, ϕ) -space and $dI = r dr dz d\phi$. Then $\lambda = -m/\rho$, where $m = \int \sigma dI$ and ρ is now distance from the origin. By (18), this gives in E the metric

$$\begin{aligned} \Phi = \exp\left(\frac{2m}{\rho} - \frac{m^2 r^2}{\rho^4}\right) (dr^2 + dz^2) \\ + r^2 \exp\left(\frac{2m}{\rho}\right) d\phi^2 - \exp\left(-\frac{2m}{\rho}\right) dt^2, \quad (30) \\ \rho^2 = r^2 + z^2. \end{aligned}$$

But what is the metric in the interior I ? This raises a broad question, for there is no well-stated problem as far as I is concerned. In VII-§ 7 the spherically symmetric fluid sphere was completely investigated, and one might think that fluid bodies should claim prior attention here. But we know from Newtonian physics that the *statical* condition limits the equilibrium-forms of fluids to spherical symmetry, and we are not to expect to find in relativity a statical fluid body which is axially symmetric without being spherically symmetric.

In fact, we must abandon fluids, and indeed abandon the idea that the structure of the matter is given a priori. We should be satisfied with any energy tensor which shows pressure rather than tension and (even more important) positive density. This means that we demand satisfaction in I of the conditions [cf. IV-(146a)]

$$T_1^1 \geq 0, \quad T_2^2 \geq 0, \quad T_3^3 \geq 0, \quad T_4^4 < 0, \quad (31)$$

or equivalently

$$G_1^1 \leq 0, \quad G_2^2 \leq 0, \quad G_3^3 \leq 0, \quad G_4^4 > 0. \quad (32)$$

We see before us a body bounded by a surface S with equation

$$f(r, z) = 0, \quad (33)$$

separating an exterior region E from an interior region I . On S the

Einstein tensor of I must satisfy the two junction conditions ¹

$$\begin{aligned} G_1^1 f_{,1} + G_1^2 f_{,2} &= 0, \\ G_2^1 f_{,1} + G_2^2 f_{,2} &= 0. \end{aligned} \quad (34)$$

To construct an axially symmetric field in E and I , we have to assign a metric (3) involving *three* functions α, β, γ of x^1, x^2 (equivalently, of r, z). In E they are to reduce to *two*, by the relation $\beta\gamma = r$ as in (13), but this reduction is not to occur in I , because it implies (15) and hence

$$G_1^1 + G_2^2 = 0; \quad (35)$$

this violates (32) unless G_1^1 and G_2^2 both vanish, which we do not desire. In both E and I , elementary flatness demands that $r\alpha/\beta \rightarrow 1$ as $r \rightarrow 0$. Since the problem in E is so well under control, and the requirements in I consist only in the boundary conditions (34) and the inequalities (32), it seems natural to start with E and work inwards. But the way is not easy. One can resolve the steps as follows:

- (i) Choose in E a harmonic function $\lambda(r, z)$ which vanishes at infinity like $(r^2 + z^2)^{-\frac{1}{2}}$.
- (ii) Calculate $\nu(r, z)$ in E by (23).
- (iii) Calculate (α, β, γ) in E by (17).
- (iv) Choose in I functions (α, β, γ) , continuous across S with (α, β, γ) in E .
- (v) Calculate by (4) R_{ij} in I , and hence G_j^i in I .
- (vi) Test by (34) and revise choices (i) and (iv) so that (34) is satisfied.
- (vii) Test by (32) throughout I .

If (32) is satisfied, then (3) is a suitable metric for a complete axially-symmetric universe, with matter in I and vacuum in E .

§ 2. SPACE-TIMES CONFORMALLY RELATED AND CONFORMALLY FLAT

The general theory of relativity is bedevilled by the large number of unknown functions — the ten components g_{ij} . There is little hope of getting physically interesting results without making a drastic reduction in their number. In Chap. VII the hypothesis of spherical

¹ Cf. I-(229) et seq. We are *not* assuming (r, ϕ, z) to be admissible coordinates, for that would be rash. The conditions (34) express the vanishing of stress across S .

symmetry reduced the number to *two*, and in § 1 a rather strengthened hypothesis of axial symmetry reduced it to *three* (*two* in vacuo). We shall now go further and reduce the number to *one* by dealing with conformally flat space-time.

But before proceeding to this, it is well to develop some results for two space-times which are *conformally related*. This means that their metric tensors, g_{ij} and g'_{ij} , satisfy a relation

$$g_{ij} = e^{\psi(x)} g'_{ij}. \quad (36)$$

We have obviously

$$g^{ij} = e^{-\psi} g'^{ij}, \quad (37)$$

and we find, for the Christoffel symbols,

$$\Gamma^i_{jk} = \Gamma'^i_{jk} + A^i_{jk} \quad (38)$$

where

$$A^i_{jk} = \frac{1}{2}(\delta^i_j \psi_{,k} + \delta^i_k \psi_{,j} - g'_{jk} g'^{ia} \psi_{,a}). \quad (39)$$

Substituting in 1-(88), we find that the two Riemann tensors are related to one another as follows:

$$R^i_{jkm} = R'^i_{jkm} + A^i_{jm|k} - A^i_{jk|m} + A^a_{jm} A^i_{ak} - A^a_{jk} A^i_{am}, \quad (40)$$

where the stroke indicates covariant differentiation with respect to g'_{ij} , not g_{ij} . (Obviously ψ is an invariant, and A^i_{jk} a tensor.)

Contraction of (40) gives

$$R_{jk} = R'_{jk} + A^a_{aj|k} - A^a_{jk|a} + A^a_{jb} A^b_{ak} - A^a_{jk} A^b_{ab}. \quad (41)$$

Now by (39)

$$\begin{aligned} A^a_{aj} &= 2\psi_{,j}, & A^a_{aj|k} &= 2\psi_{|jk}, & A^a_{jk|a} &= \psi_{|jk} - \frac{1}{2}g'_{jk} \square' \psi, \\ A^a_{jb} A^b_{ka} &= \frac{3}{2}\psi_{,j}\psi_{,k} - \frac{1}{2}g'_{jk}\chi, & A^a_{jk} A^b_{ab} &= 2\psi_{,j}\psi_{,k} - g'_{jk}\chi, \end{aligned} \quad (42)$$

where

$$\chi = g'^{ab}\psi_{,a}\psi_{,b}, \quad \square' \psi = g'^{ab}\psi_{|ab}. \quad (43)$$

Thus the relation between the Ricci tensors is

$$R_{jk} = R'_{jk} + \psi_{|jk} - \frac{1}{2}\psi_{,j}\psi_{,k} + \frac{1}{2}g'_{jk}(\square' \psi + \chi). \quad (44)$$

For the curvature invariants we have

$$R = g^{jk}R_{jk} = e^{-\psi}g'^{jk}R_{jk} = e^{-\psi}(R' + 3\square' \psi + \frac{3}{2}\chi). \quad (45)$$

Finally for the Einstein tensors we get

$$\begin{aligned} G_{jk} &= R_{jk} - \frac{1}{2}g_{jk}R = R_{jk} - \frac{1}{2}e^\psi g'_{jk}R \\ &= G'_{jk} + \psi_{|jk} - \frac{1}{2}\psi_{,j}\psi_{,k} - g'_{jk}(\square'\psi + \frac{1}{4}\chi). \end{aligned} \quad (46)$$

This formula enables us to transform one gravitational problem into another. Suppose we have a field g'_{ij} with matter represented by G'_{ij} . Then, choosing any function ψ , we get a new field with g_{ij} as in (36) and G_{ij} as in (47). Such a transformation is of course something quite different from a mere transformation of coordinates, for the latter is nothing but a formal change in the way a certain field is expressed mathematically.

If g'_{ij} is the metric of flat space-time, the space-time with metric g_{ij} as in (36) is said to be *conformally flat*. We can then delete $R'^i{}_{jkm}$, R'_{jk} , G'_{jk} in the above equations; in particular we have

$$G_{jk} = \psi_{|jk} - \frac{1}{2}\psi_{,j}\psi_{,k} - g'_{jk}(\square'\psi + \frac{1}{4}\chi). \quad (47)$$

If we like, we can use coordinates such that

$$g'_{ij} = \eta_{ij} = \text{diag}(1, 1, 1, -1); \quad (48)$$

then

$$\begin{aligned} g_{ij} &= e^\psi \eta_{ij}, \\ \Phi &= e^\psi [(dx^1)^2 + (dx^2)^2 + (dx^3)^2 - (dx^4)^2]. \end{aligned} \quad (49)$$

The covariant derivatives in (47) may be replaced by partial derivatives, so that we have

$$\begin{aligned} G_{ij} &= \psi_{,ij} - \frac{1}{2}\psi_{,i}\psi_{,j} - \eta_{ij}(\square\psi + \frac{1}{4}\chi), \\ \square\psi &= \eta^{ab}\psi_{,ab}, \quad \chi = \eta^{ab}\psi_{,a}\psi_{,b}. \end{aligned} \quad (50)$$

The metric g_{ij} depends solely on the function ψ , and, by making suitable choice of this function, we can generate a multiplicity of universes in which the energy tensor T_{ij} is given by

$$\kappa T_{ij} = -G_{ij}, \quad \kappa = 8\pi, \quad (51)$$

with G_{ij} as in (50). The physical interest of such universes will, however, depend on whether they exhibit pressure (rather than tension) and positive energy density — in fact we desire to satisfy the inequalities (32).

Let us examine the case where ψ is a function of x^4 only. Denoting its derivatives by primes, we have

$$\begin{aligned}\psi_{,\alpha} &= 0, & \psi_{,4} &= \psi', & \psi_{,44} &= \psi'', \\ \square\psi &= -\psi'', & \chi &= -\psi'^2.\end{aligned}\tag{52}$$

Then (50) gives for the Einstein tensor ¹

$$\begin{aligned}G_{\alpha\beta} &= \delta_{\alpha\beta}(\psi'' + \tfrac{1}{4}\psi'^2), \\ G_{\alpha 4} &= 0, \\ G_{44} &= -\tfrac{3}{4}\psi'^2.\end{aligned}\tag{53}$$

The inequalities (32) are satisfied provided

$$\psi'' + \tfrac{1}{4}\psi'^2 \leq 0.\tag{54}$$

In fact, (53) gives a positive energy-density, but tension rather than pressure unless (54) holds.

It is impossible to satisfy (54) in the whole range of x^4 . But if (writing $x^4 = t$) we are content to deal only with $t > 0$ or $t < 0$, we get an interesting universe on putting

$$e^\psi = \left(\frac{t}{a}\right)^4\tag{55}$$

where a is any constant; the metric form may now be written

$$\Phi = \left(\frac{t}{a}\right)^4 (dx^2 + dy^2 + dz^2 - dt^2).\tag{56}$$

We have

$$\begin{aligned}\psi' &= \frac{4}{t}, & \psi'' &= -\frac{4}{t^2}, & \psi'' + \tfrac{1}{4}\psi'^2 &= 0, \\ G_{\alpha\beta} &= 0, & G_{44} &= -\frac{12}{t^2}.\end{aligned}\tag{57}$$

¹ When ψ is a function of x^4 only, we have a universe with spherical symmetry in the sense of VII-§ 3, and the formulae of that section might also be used; we may check the values of G_{ij} in (53) by putting in VII-(78) $\alpha = \gamma = \psi$, $\beta = \psi + 2 \log r$, and taking ψ to be a function of x^4 only.

In this universe there is no stress, no flux of energy, and a positive energy-density

$$\mu = -T_4^4 = \kappa^{-1}G_4^4 = \frac{12a^4}{\kappa t^6}, \quad (58)$$

which tends to infinity as $t \rightarrow 0$. It is the universe of EINSTEIN and DE SITTER [1932].

Another interesting choice is

$$e^\psi = \left(\frac{a}{t}\right)^2, \quad \Phi = \left(\frac{a}{t}\right)^2 (dx^2 + dy^2 + dz^2 - dt^2), \quad (59)$$

for which

$$\begin{aligned} \psi' &= -\frac{2}{t}, & \psi'' &= \frac{2}{t^2}, & \psi'' + \frac{1}{4}\psi'^2 &= \frac{3}{t^2}, \\ G_{\alpha\beta} &= \delta_{\alpha\beta} \frac{3}{t^2}, & G_{\alpha 4} &= 0, & G_{44} &= -\frac{3}{t^2}, \end{aligned} \quad (60)$$

and so

$$G_{ij} = \Lambda g_{ij}, \quad \Lambda = 3a^{-2}. \quad (61)$$

The inequality (54) is violated, but here we do not care, because (32) (and hence (54)) assumed that the cosmological constant Λ is zero. Reinstating it, and turning back to VII-(4), we see that in (59) we have recreated the de Sitter universe with constant curvature a^{-2} . The moral of this is that the same universe may appear in many different guises according to the coordinates used, and that one should concentrate on invariant properties. Putting

$$\tau/a = \log(t/a), \quad (62)$$

we can change the metric form of the de Sitter universe to ¹

$$\Phi = e^{-2\tau/a}(dx^2 + dy^2 + dz^2) - d\tau^2. \quad (63)$$

It may be noted that any form

$$\Phi = e^{h(\tau)}(dx^2 + dy^2 + dz^2) - d\tau^2 \quad (64)$$

can be put into conformally flat form; for by the transformation

$$t = \int e^{-\frac{1}{2}h(\tau)} d\tau, \quad (65)$$

we have

$$d\tau^2 = e^{h(\tau)} dt^2, \quad (66)$$

¹ Cf. ROBERTSON [1933] for the history of this and other forms.

and (64) becomes

$$\Phi = e^{\psi(t)}(dx^2 + dy^2 + dz^2 - dt^2). \quad (67)$$

where

$$\psi(t) = h(\tau). \quad (68)$$

§ 3. THE COSMOLOGICAL RED-SHIFT

It is observed that the spectra of nebulae are shifted towards the red, the shift being roughly proportional to distance. The endurance of Newtonian concepts is such that many astronomers seem to accept absolute space and time, and attribute the red-shift of the spectra of nebulae to a velocity of recession in the Newtonian sense.

A relativist cannot, of course, look at the phenomenon of red-shift in that way. If he feels that the matter in the universe is too thinly spread to produce a significant curvature of space-time, he may use the special theory (flat space-time) ¹. But if the gravity of matter is significant, then allowance must be made for it. Without seeking the maximum generality, even under the assumptions of symmetry usually made ², we shall here assume the metric form of space-time to be conformally flat as in (49) and we shall write it

$$\Phi = [\omega(t)]^2(dx^2 + dy^2 + dz^2 - dt^2); \quad (69)$$

we shall make no specific assumption regarding the function $\omega(t)$ except that it increases with t , although we shall later put $\omega = t^2/a^2$ as in (56), since that corresponds to an absence of stress. As is always done in such arguments, we shall assume (in spite of the presence of matter due to $G_{ij} \neq 0$) that the world-line of a photon is a null geodesic with the 4-momentum tangent to it and carried by parallel transport, so that the methods of III-§ 7 and VII-§ 9 can be used.

Since $G_{\alpha 4} = 0$ in (53), the timelike eigenvector of G_{ij} points along the t -lines, which are therefore the world-lines of the matter. When we take a source (nebula) and observer, we shall assume both their world-lines to be t -lines (we shall see below that they are geodesics).

Before proceeding to the red-shift, we note that (69) represents an *expanding universe* in a real sense. For, as in III-§ 5, we can measure optically the 'distance' between two adjacent t -lines, and it increases

¹ KERMACK and MCCREA [1933], MILNE [1935], SYNGE [1956a].

² For relativistic cosmology, see ROBERTSON [1933], TOLMAN [1934b], BONDI [1952], McVITTIE [1956], HECKMANN and SCHÜCKING [1959].

with t , the value being

$$D = \omega(t)(dx^2 + dy^2 + dz^2)^{\frac{1}{2}}. \quad (70)$$

The 'distance' between two t -lines which are not adjacent is a matter to be discussed later.

The geodesics for (69) are simple to deal with. Writing the Lagrangian

$$F = \frac{1}{2}[\omega(t)]^2(x'^2 + y'^2 + z'^2 - t'^2), \quad (71)$$

the prime indicating d/dw with w a special parameter, the first three Lagrangian equations give

$$\omega^2 x' = A, \quad \omega^2 y' = B, \quad \omega^2 z' = C, \quad (72)$$

where A, B, C are constants. Therefore the geodesics are straight lines in an auxiliary Euclidean space E_3 in which (x, y, z) are rectangular Cartesians. We also see that the t -lines are timelike geodesics, the proper time on them being given by

$$ds = \omega dt. \quad (73)$$

On account of the simplicity of the form (69) in (x, y, z) , we can adequately explore all timelike and null geodesics by attending to those which pass inwards through the origin of (x, y, z) . We write

$$r = (x^2 + y^2 + z^2)^{\frac{1}{2}}, \quad (74)$$

and call r the *pseudo-distance* of (x, y, z) from the origin; it is to be regarded as a purely mathematical construct, unconnected so far with any physical measurements. For such a radial geodesic, (72) gives

$$\omega^2 dr = -k dw, \quad (75)$$

where k is a positive constant depending on the particular geodesic; further we have

$$\omega^2(dr^2 - dt^2) = -\eta dw^2, \quad (76)$$

where, for a timelike geodesic we put $\eta = 1$, $dw = ds$, and for a null geodesic, $\eta = 0$. Hence

$$\begin{aligned} dw &= \frac{\omega^2 dt}{\sqrt{\eta\omega^2 + k^2}}, \\ dr &= -k \frac{dw}{\omega^2} = -\frac{k dt}{\sqrt{\eta\omega^2 + k^2}}, \end{aligned} \quad (77)$$

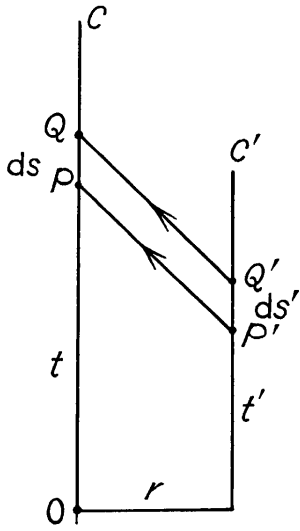


Fig. 3 — Cosmological red-shift pictured in E_4

and in particular for a null geodesic

$$dw = k^{-1}\omega^2 dt, \quad dr = -dt. \quad (78)$$

To discuss the question of red-shift, it is well to picture events in E_4 , a Euclidean 4-space with (x, y, z, t) rectangular Cartesians (Fig. 3). We see two world-lines, C and C' , representing respectively the histories of an observer and a source; C is the t -axis, and C' a t -line at pseudo-distance r . In the physical interpretation we put the observer on the sun¹ and the source on a distant nebula. Fig. VII-10 (p. 299) was much the same as Fig. 3, but whereas the former with its curved lines was only a guide to thought, the latter is a scale drawing with null

lines inclined at an angle of 45° .

In Fig. 3 $P'P$ and $Q'Q$ are the adjacent histories of two photons. If t, t' are values at P, P' , respectively, we have by (78)

$$r = t - t', \quad (79)$$

and, passing to $Q'Q$,

$$dt = dt', \quad (80)$$

where dt refers to PQ and dt' to $P'Q'$. By (73) the corresponding increments in proper time are

$$ds = \omega(t)dt, \quad ds' = \omega(t')dt', \quad (81)$$

and III-(52) gives a spectral shift

$$\frac{\nu' - \nu}{\nu'} = \frac{ds - ds'}{ds} = 1 - \frac{\omega(t')}{\omega(t)}. \quad (82)$$

Thus the metric form (69) gives a red-shift on the assumption that $\omega(t)$ is an increasing function.

We have now to inquire how the red-shift depends on the distance of the source, and for that the concept of 'distance' must be defined, for we can hardly attach physical meaning to the pseudo-distance r .

¹ The gravitational field of the sun is not considered. In accepting (69), we smeared matter out into a thin uniform distribution, in accordance with the common practice in treating such cosmological problems.

Fig. 4, drawn in E_3 , shows at C' a source which emits radiation uniformly in all directions with intensity I' , by which we mean that the total energy emitted in proper time ds' is $I'ds'$. There is a receiver or observer at C . The points C, C' in Fig. 4 are the projections on E_3 of the lines C, C' of Fig. 3. The energy $I'ds'$ is of course computed relative to the world-line C' .

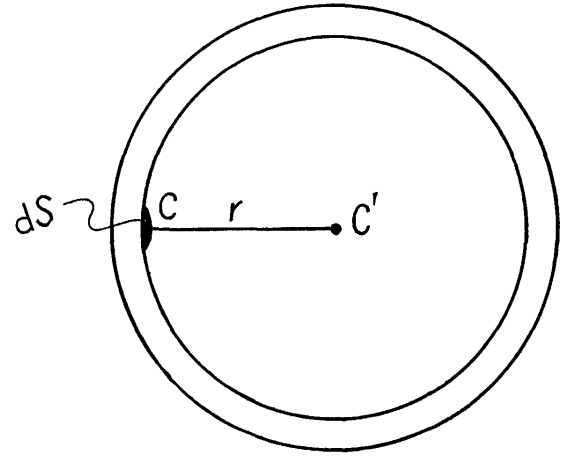


Fig. 4 – Spherical shell of radiation emitted from C'

The energy, relative to a world-line with 4-velocity V^i , of a photon with 4-momentum p^i is $E = -p^i V_i$. We are concerned only with t -lines, so that V^4 is the only surviving component of V^i , and

$$V^4 = \omega^{-1}, \quad V_4 = -\omega. \quad (83)$$

The 4-momentum of the photon may be written in general

$$p^i = \alpha \frac{dx^i}{dw}, \quad (84)$$

where w is a special parameter on its world-line and α a constant depending on the choice of w and the particular photon involved. In our problem we have, by (78),

$$p^4 = \alpha \frac{dt}{dw} = \frac{\alpha k}{\omega^2}, \quad (85)$$

and so the energy of the photon is

$$E = \frac{\alpha k}{\omega}. \quad (86)$$

Since $\omega(t)$ is an increasing function, by assumption, the photon continually loses energy, relative to the local t -line. As a photon passes from emission at C' to reception at C , the energy is decreased according to the formula

$$\frac{E}{E'} = \frac{\omega(t')}{\omega(t)}. \quad (87)$$

This confirms, by a different method, the red-shift already found in (82).

When the burst of energy $I'ds'$, emitted at t' from C' , reaches C at t , it occupies in Fig. 4 a thin spherical shell, and the total energy in this shell is, by (87),

$$I'ds' \frac{\omega(t')}{\omega(t)}, \quad (88)$$

the local energies over the sphere being added together.

At C the observer sets up a target (actually the object glass of a telescope) to catch the radiation. If dS is the (invariant) area of the target and ds the increment in the observer's proper time from the beginning to the end of the burst, then the total energy received by him may be written

$$JdSds, \quad (89)$$

J being the intensity of the received radiation.

Now the pseudo-radius of the sphere and its invariant area are

$$r = t - t', \quad dS = 4\pi r^2 [\omega(t)]^2, \quad (90)$$

and so C receives on his target a fraction

$$\frac{dS}{4\pi r^2 [\omega(t)]^2} \quad (91)$$

of all the energy in the shell. Thus

$$JdSds = I'ds' \frac{dS}{4\pi r^2} \frac{\omega(t')}{[\omega(t)]^3}. \quad (92)$$

But as in (82)

$$ds' = ds \frac{\omega(t')}{\omega(t)}, \quad (93)$$

and so the received intensity is

$$J = \frac{I'}{4\pi r^2} \frac{[\omega(t')]^2}{[\omega(t)]^4}, \quad (94)$$

$$r = t - t'.$$

The ratio J/I' is therefore the same for all sources of different brightness at the same pseudo-distance r , and we may define the *astro-*

*nomical*¹ distance r_0 by the formula

$$J = \frac{I'}{4\pi r_0^2}, \quad (95)$$

which makes $r_0 = r$ in flat space-time. If we assume, as astronomers do, the existence throughout the universe of stars which all have the same intrinsic luminosity I' , the measurement of the intensity J received from such a star gives the astronomical distance r_0 of that star and of any others known to be near to it. In terms of r_0 , (94) reads

$$r_0 = (t - t') \frac{[\omega(t)]^2}{\omega(t')}. \quad (96)$$

Denoting the red-shift as in (82) by ρ , we have

$$\rho = \frac{\nu' - \nu}{\nu'} = 1 - \frac{\omega(t')}{\omega(t)}. \quad (97)$$

This equation and (96) are basic. The left hand sides are observational, and the right hand sides are theoretical constructs. If observation gives a relationship between ρ and r_0 , these equations lead to a functional equation for the function ω . On the other hand, if we assume a function ω , elimination of t' will give a relationship between ρ and r_0 .

If, to explore the situation, we assume $(t - t')$ small, we may write approximately

$$\omega(t') = \omega(t) - (t - t')\omega'(t); \quad (98)$$

then we get

$$\begin{aligned} r_0 &= (t - t')\omega(t), & \rho &= (t - t')\omega'(t), \\ \rho &= r_0 \frac{\omega'(t)}{\omega(t)}, \end{aligned} \quad (99)$$

so that red-shift is proportional to optical distance for all observations made at t .

If, scorning approximations, we take the Einstein-de Sitter form (56),

$$\Phi = \left(\frac{t}{a}\right)^4 (dx^2 + dy^2 + dz^2 - dt^2), \quad (100)$$

¹ Or *luminosity distance*. This is a particular application of the definition of *spatial distance* given by WHITTAKER [1931].

so that

$$\omega(t) = \left(\frac{t}{a}\right)^2, \quad (101)$$

and the density is, as in (58),

$$\mu = \frac{12a^4}{\kappa t^6}, \quad (102)$$

then (96) and (97) give

$$r_0 = (t - t') \frac{t^4}{a^2 t'^2}, \quad \rho = 1 - \frac{t'^2}{t^2}. \quad (103)$$

Eliminating t' , we get the following shift-distance law for observations made at t :

$$r_0 = \frac{t^3}{a^2} \cdot \frac{1 - \sqrt{1 - \rho}}{1 - \rho}. \quad (104)$$

So far, no approximation. But now expand in powers of ρ ,

$$r_0 = \frac{1}{2} \frac{t^3}{a^2} \left(\rho + \frac{5}{4} \rho^2 + \dots \right), \quad (105)$$

and cut this off at the first term, obtaining the linear law

$$\frac{\rho}{r_0} = \frac{2a^2}{t^3}. \quad (106)$$

The observational quantity

$$\sigma = \frac{\rho}{r_0} \quad (107)$$

is called *Hubble's constant* (or parameter), and its value is roughly ¹

$$\sigma = 4 \times 10^{-18} \text{ sec}^{-1}. \quad (108)$$

Then (106) gives

$$\frac{t^3}{a^2} = 5 \times 10^{17} \text{ sec}. \quad (109)$$

¹ Cf. McVITTIE [1956, p. 167], where this parameter is denoted by h_1 . There is a considerable range of uncertainty in the value on account of the difficulty in estimating r_0 . The value quoted in SYNGE [1956a] was about twice the figure given in (108).

If we put $t = 0$ in (100), the metric collapses to zero, and this may be interpreted as the beginning of the universe. Thus the age of the universe, up to t , is

$$\int_{t=0}^t ds = \int_{t=0}^t \omega(t) dt = \int_{t=0}^t \frac{t^2}{a^2} dt = \frac{1}{3} \frac{t^3}{a^2}. \quad (110)$$

Inserting the value (109), and noting that 1 year = 3.1558×10^7 sec, we get, for the age of the universe ¹,

$$1.7 \times 10^{17} \text{ sec} = 5.3 \times 10^9 \text{ years}, \quad (111)$$

measured in proper time.

We turn now to (102) for an estimation of density, which becomes infinite for $t = 0$ and is, for a general t ,

$$\mu = \frac{3}{2\pi} \frac{a^4}{t^6} = \frac{3\sigma^2}{8\pi}. \quad (112)$$

With the value (108) for σ this gives ²

$$\mu = 1.910 \times 10^{-36} \text{ sec}^{-2}. \quad (113)$$

Using the conversion formulae

$$\begin{aligned} 1 \text{ g} &= 2.476 \times 10^{-39} \text{ sec}, \\ 1 \text{ cm}^{-1} &= 2.998 \times 10^{10} \text{ sec}^{-1}, \\ 1 \text{ g cm}^{-3} &= 6.668 \times 10^{-8} \text{ sec}^{-2}, \\ 1 \text{ sec}^{-2} &= 1.500 \times 10^7 \text{ g cm}^{-3}, \end{aligned} \quad (114)$$

we get a density ³

$$\mu = 2.865 \times 10^{-29} \text{ g cm}^{-3}. \quad (115)$$

Of all branches of modern science, cosmological theory is the least

¹ Note that in this work the age of the universe is not the reciprocal of Hubble's constant, but $2/3$ of the reciprocal.

² For purposes of arithmetical calculation, it is unwise to round off numbers, even if their values are physically uncertain.

³ This value is about six times the upper bound given by McVITTIE [1956, p. 176], who allows in his range a factor of 100. BONDI [1952, pp. 46, 48] gave a much larger value of $10^{-27} \text{ g cm}^{-3}$.

disciplined by observation. Optical observations made at any instant on the world-line C (Fig. 5) have their sources in events lying on the null cone drawn into the past, and all man's observations over a period of, say, a thousand years come from a null shell, which, on the cosmical time-scale, must be regarded as wafer-thin. From this small sample of the universe and from geological history on the world-line C itself, man attempts to construct the whole by daring extrapolation. Since we cannot dispute about the unknowable, any theory is successful if it succeeds in the thin shell of the known. But unfortunately the interpretation of observations of this shell is difficult, and

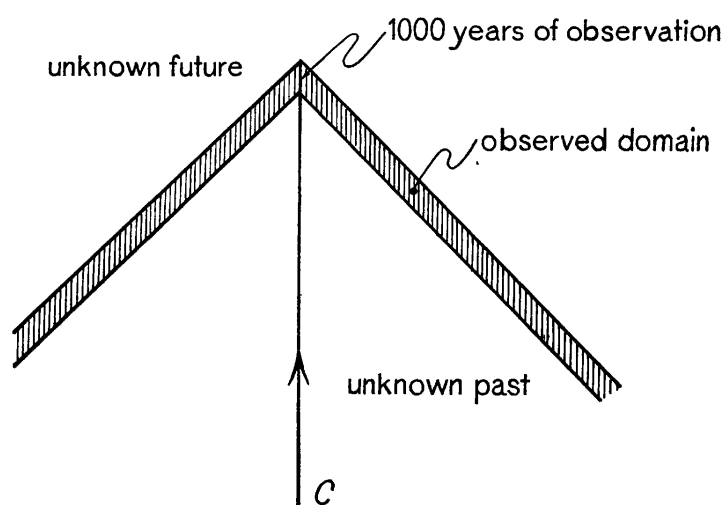


Fig. 5 – The thin null shell formed by observed events

much must be left to the inspired guesses of astronomers who change their verdicts from year to year. All this should not deter us from the creation of cosmological models, but it does suggest that the rival merits of different models should not be supported with polemical heat.

Simplicity is a grand thing, and we may well prefer the simplest model of the universe. As an exercise on conformally flat universes, we selected first one in which the conformal factor was a function of t only, and then specialized to the form (100) which gives zero pressure without the intervention of the cosmological constant. This model universe, invented by EINSTEIN and DE SITTER [1932]¹, seems to be the simplest there can be, and deserves attention; it is not suggested that it necessarily represents physical reality as well as, or better than, more complicated models.

¹ Cf. TOLMAN [1934b, p. 415], who gives an account of many model universes, in most of which the cosmological constant is involved.

§ 4. UNIVERSES OF THE GÖDEL TYPE

In building a model universe we choose a metric form $g_{ij}dx^i dx^j$ and apply certain tests to it. First, it must be of signature ¹ $+2$. Secondly, stress and density must be explored, for we wish stress to be pressure rather than tension, and density to be positive ². Since the decision here rests merely on signs and it is easy to get confused, let us restate in a slightly different way the test already described in IV-§ 6.

To get pressure rather than tension ³ and a positive density μ , we require that the determinantal equation

$$\det(G_{ij} - \theta g_{ij}) = 0 \quad (116)$$

should have *three negative roots and one positive root*. This positive root is $\kappa\mu$ ($= 8\pi\mu$). It is further required that the unit vector V^i (4-velocity) satisfying

$$G_{ij}V^j = \kappa\mu g_{ij}V^j \quad (117)$$

should be *timelike* ($V_i V^i = -1$). As a simple check on this, we write down formulae for a perfect fluid, W^i being any vector orthogonal to the 4-velocity V^i :

$$\begin{aligned} T_{ij} &= (\mu + p)V_i V_j + p g_{ij}, \\ G_{ij} &= -\kappa(\mu + p)V_i V_j - \kappa p g_{ij}, \\ G_{ij}W^j &= -\kappa p g_{ij}W^j, \\ G_{ij}V^j &= \kappa\mu g_{ij}V^j. \end{aligned} \quad (118)$$

A strange and interesting model universe was invented by GÖDEL [1949], [1950]. We shall now explore a type of metric which includes his as a particular case.

¹ The signature of the quadratic form is the number of positive terms less the number of negative terms which occur when, by local transformation of coordinates, the matrix g_{ij} is made diagonal. In this book the signature $+2$ is used, but many writers prefer -2 . In any particular case, one can pass from the one to the other by reversing the signs of all the components g_{ij} ; this makes no physical difference whatever in the universe under consideration, but it does lead to confusing changes of sign in certain formulae (see Appendix A). A further source of confusion in the comparison of formulae arises from the fact that while most writers define R_{ij} as in I-(105), others define it as the negative of this.

² The cosmological constant affects both stress and density; in this work we shall put $\Lambda = 0$.

³ We may conveniently speak of this as *positive pressure*, without implying that the pressure is *isotropic*.

Throughout this section, Greek suffixes take the values 1, 2, and capital suffixes the values 3, 4.

Consider a metric form

$$\Phi = g_{\alpha\beta} dx^\alpha dx^\beta + g_{AB} dx^A dx^B, \quad (119)$$

where $g_{\alpha\beta}$ are functions of (x^3, x^4) and g_{AB} are constants. Although much more general than the form proposed by him, we may refer to this as of Gödel type. The correct signature for space-time is assured if, for the two quadratic forms in (119), we take one of signature 0 and the other signature $+2$, in either order.

For (119) we have

$$g_{\alpha B} = 0, \quad g^{\alpha B} = 0, \quad g_{\alpha\beta} g^{\beta\gamma} = \delta_\alpha^\gamma, \quad g_{AB} g^{BC} = \delta_A^C. \quad (120)$$

To survive, a Christoffel symbol must have precisely two Greek letters; thus the survivors are

$$[\alpha\beta, C] = -\frac{1}{2}g_{\alpha\beta,C}, \quad [\alpha C, \beta] = \frac{1}{2}g_{\alpha\beta,C}, \quad (121)$$

and 1-(85) gives, as surviving components of the Riemann tensor,

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= \frac{1}{4}g^{EF}(g_{\alpha\delta,E}g_{\beta\gamma,F} - g_{\alpha\gamma,E}g_{\beta\delta,F}), \\ R_{\alpha\beta CD} &= \frac{1}{4}g^{\gamma\delta}(g_{\alpha\gamma,D}g_{\beta\delta,C} - g_{\alpha\gamma,C}g_{\beta\delta,D}), \\ R_{\alpha B\gamma D} &= -\frac{1}{2}g_{\alpha\gamma,BD} + \frac{1}{4}g^{\rho\sigma}g_{\alpha\rho,D}g_{\gamma\sigma,B}. \end{aligned} \quad (122)$$

At this point we specialize by assuming $g_{\alpha\beta}$ to be functions of x^4 only. Then $R_{\alpha\beta CD}$ vanishes, and the surviving independent components in (122) may be written out as follows:

$$\begin{aligned} R_{1212} &= \frac{1}{4}g^{44}[(g_{12,4})^2 - g_{11,4}g_{22,4}], \\ R_{1414} &= -\frac{1}{2}g_{11,44} + \frac{1}{4}g^{\rho\sigma}g_{\rho 1,4}g_{\sigma 1,4}, \\ R_{1424} &= -\frac{1}{2}g_{12,44} + \frac{1}{4}g^{\rho\sigma}g_{\rho 1,4}g_{\sigma 2,4}, \\ R_{2424} &= -\frac{1}{2}g_{22,44} + \frac{1}{4}g^{\rho\sigma}g_{\rho 2,4}g_{\sigma 2,4}. \end{aligned} \quad (123)$$

As a further drastic specialization suggested by the work of Gödel we choose

$$g_{\alpha\beta} = \begin{pmatrix} a & be^\psi \\ be^\psi & ce^{2\psi} \end{pmatrix}, \quad (124)$$

where a, b, c are constants and ψ any function of x^4 (we shall indicate its derivatives by primes), while for g_{AB} we choose any one of the following three matrices:

$$g_{AB} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (125)$$

The particular choice must be related to the choice of a, b, c , in order to secure the correct signature.

Write

$$\sigma = b^2 - ac, \quad \varepsilon = \det g_{AB} = \pm 1. \quad (126)$$

Then

$$\begin{aligned} g &= -\varepsilon\sigma e^{2\psi}, \\ g^{\alpha\beta} &= \sigma^{-1} \begin{pmatrix} -c & be^{-\psi} \\ be^{-\psi} & -ae^{-2\psi} \end{pmatrix}, \\ g^{AB} &= g_{AB}. \end{aligned} \quad (127)$$

The signature requirement is either

$$\sigma > 0, \quad \varepsilon = 1, \quad (128)$$

or

$$\sigma < 0, \quad \varepsilon = -1, \quad a > 0. \quad (129)$$

The former makes $g_{\alpha\beta}dx^\alpha dx^\beta$ indefinite and $g_{AB}dx^A dx^B$ positive-definite, and the latter reverses this.

From (123) we obtain the following simple formulae:

$$\begin{aligned} R_{1212} &= \frac{1}{4}g^{44}b^2\psi'^2 e^{2\psi}, \\ R_{1414} &= -\frac{1}{4} \frac{ab^2}{\sigma} \psi'^2, \\ R_{1424} &= -\frac{1}{2}b \left[\psi'' + \frac{b^2}{2\sigma} \psi'^2 \right] e^\psi, \\ R_{2424} &= -c \left[\psi'' + \left(1 + \frac{b^2}{4\sigma} \right) \psi'^2 \right] e^{2\psi}. \end{aligned} \quad (130)$$

Hence for the surviving components of the Ricci tensor and for the

curvature invariant we have

$$\begin{aligned}
 R_{11} &= g^{\alpha\beta} R_{\alpha 11 \beta} + g^{44} R_{4114} = \frac{1}{2} g^{44} \frac{ab^2}{\sigma} \psi'^2, \\
 R_{12} &= g^{\alpha\beta} R_{\alpha 12 \beta} + g^{44} R_{4124} = \frac{1}{2} g^{44} b \left(\psi'' + \frac{b^2}{\sigma} \psi'^2 \right) e^\psi, \\
 R_{22} &= g^{\alpha\beta} R_{\alpha 22 \beta} + g^{44} R_{4224} = g^{44} c \left[\psi'' + \left(1 + \frac{b^2}{2\sigma} \right) \psi'^2 \right] e^{2\psi}, \\
 R_{44} &= g^{\alpha\beta} R_{\alpha 44 \beta} = \psi'' + \left(1 - \frac{b^2}{2\sigma} \right) \psi'^2, \\
 R &= 2g^{44} \left[\psi'' + \left(1 - \frac{b^2}{4\sigma} \right) \psi'^2 \right].
 \end{aligned} \tag{131}$$

Calculating the Einstein tensor by

$$G_{ij} = R_{ij} - \frac{1}{2} g_{ij} R, \tag{132}$$

we obtain the following surviving components:

$$\begin{aligned}
 G_{11} &= -ag^{44} \left[\psi'' + \left(1 - \frac{3b^2}{4\sigma} \right) \psi'^2 \right], \\
 G_{12} &= -bg^{44} \left[\frac{1}{2} \psi'' + \left(1 - \frac{3b^2}{4\sigma} \right) \psi'^2 \right] e^\psi, \\
 G_{22} &= cg^{44} \frac{3b^2}{4\sigma} \psi'^2 e^{2\psi}, \\
 G_{33} &= -g_{33} g^{44} \left[\psi'' + \left(1 - \frac{b^2}{4\sigma} \right) \psi'^2 \right], \\
 G_{44} &= -\frac{b^2}{4\sigma} \psi'^2.
 \end{aligned} \tag{133}$$

This is the Einstein tensor for the form

$$\Phi = a(dx^1)^2 + 2be^\psi dx^1 dx^2 + ce^{2\psi} (dx^2)^2 + g_{33} (dx^3)^2 + g_{44} (dx^4)^2, \tag{134}$$

so far without restriction on the function $\psi(x^4)$ or on the constants a, b, c, g_{33}, g_{44} except for (125); the calculations did not actually involve (128) or (129).

Since $G_{\alpha A} = 0, g_{\alpha A} = 0$, two of the eigenvectors λ^i of G_{ij} and the

corresponding eigenvalues θ satisfy

$$\begin{aligned}(G_{11} - \theta g_{11})\lambda^1 + (G_{12} - \theta g_{12})\lambda^2 &= 0, \\ (G_{21} - \theta g_{21})\lambda^1 + (G_{22} - \theta g_{22})\lambda^2 &= 0, \\ \lambda^3 &= \lambda^4 = 0;\end{aligned}\tag{135}$$

these eigenvectors lie in the 2-element containing the parametric lines of x^1 and x^2 . The third eigenvector is in the x^3 -direction, and the eigenvalue is

$$\theta_3 = -g^{44} \left[\psi'' + \left(1 - \frac{b^2}{4\sigma} \right) \psi'^2 \right]; \quad (136)$$

the fourth eigenvector is in the x^4 -direction with eigenvalue

$$\theta_4 = -g^{44} \frac{b^2}{4\sigma} \psi'^2. \quad (137)$$

If we choose at random the function $\psi(x^4)$ and the various constants, we shall probably violate the signature condition and the conditions attached to (116) for positive pressure and density. The situation is really too complicated to explore systematically, and it is necessary to choose some special form for $\psi(x^4)$. One feasible choice is

$$e^\psi = \left(\frac{x^4}{k} \right)^n \quad (k, n = \text{const.}), \quad (138)$$

but we shall follow Gödel in making a still simpler choice:

$$\psi = kx^4 \quad (k = \text{const.}). \quad (139)$$

Then $\psi' = k$, $\psi'' = 0$, and (133) simplify somewhat to

$$\begin{aligned}G_{11} &= -k^2 a g^{44} \left(1 - \frac{3b^2}{4\sigma} \right), \\ G_{12} &= -k^2 b g^{44} \left(1 - \frac{3b^2}{4\sigma} \right) e^\psi, \\ G_{22} &= k^2 c g^{44} \frac{3b^2}{4\sigma} e^{2\psi}, \\ G_{33} &= -k^2 g_{33} g^{44} \left(1 - \frac{b^2}{4\sigma} \right), \\ G_{44} &= -k^2 \frac{b^2}{4\sigma}.\end{aligned}\tag{140}$$

Now we have

$$\frac{G_{11}}{g_{11}} = \frac{G_{12}}{g_{12}}, \quad (141)$$

and so by (135) the first eigenvector is in the x^1 -direction and the eigenvalue is

$$\theta_1 = -k^2 g^{44} \left(1 - \frac{3b^2}{4\sigma} \right). \quad (142)$$

The second eigenvector does not point in the x^2 -direction, but nevertheless we shall denote its eigenvalue by θ_2 . By (135) it satisfies the determinantal equation

$$\begin{vmatrix} a & b \\ b \left[-k^2 g^{44} \left(1 - \frac{3b^2}{4\sigma} \right) - \theta_2 \right] & c \left(k^2 g^{44} \frac{3b^2}{4\sigma} - \theta_2 \right) \end{vmatrix} = 0, \quad (143)$$

and this gives

$$\theta_2 = -k^2 g^{44} \frac{b^2}{4\sigma}. \quad (144)$$

By (136) and (137) the other eigenvalues are

$$\theta_3 = -k^2 g^{44} \left(1 - \frac{b^2}{4\sigma} \right), \quad (145)$$

$$\theta_4 = -k^2 g^{44} \frac{b^2}{4\sigma}. \quad (146)$$

To obtain positive pressure and density, we are to have three eigenvalues negative (with spacelike eigenvectors) and one eigenvalue positive (with timelike eigenvector). Since $\theta_2 = \theta_4$, their common value must be negative and their eigenvectors spacelike. Thus the x^4 -direction must be spacelike, and hence

$$g^{44} = g_{44} = 1, \quad \sigma > 0. \quad (147)$$

By the signature condition (128) then it follows that $\varepsilon = 1$ and so

$$g_{AB} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad (148)$$

only the first choice in (125) is possible. Now $g_{33} = 1$ and this implies that the x^3 -direction is spacelike; its eigenvalue is therefore a stress,

and application of the positive-pressure condition to (145) give

$$1 - \frac{b^2}{4\sigma} > 0. \quad (149)$$

We have now three spacelike eigenvectors corresponding to $(\theta_2, \theta_3, \theta_4)$, and the other eigenvector must be timelike, being orthogonal to these. But it points in the x^1 -direction, and so

$$a < 0. \quad (150)$$

Further applying to (142) the condition of positive density, we get

$$1 - \frac{3b^2}{4\sigma} < 0. \quad (151)$$

Since $\sigma = b^2 - ac$, the above inequalities are equivalent to

$$a < 0, \quad c < 0, \quad \frac{4}{3}ac < b^2 < 4ac. \quad (152)$$

To sum up at this point, for $\psi = kx^4$, the only admissible form of the type (134) is

$$\Phi = a(dx^1)^2 + 2be^{kx^4}dx^1dx^2 + ce^{2kx^4}(dx^2)^2 + (dx^3)^2 + (dx^4)^2, \quad (153)$$

with a, b, c arbitrary constants subject to the inequalities (152); the principal stresses and density are

$$\begin{aligned} S_2 = S_4 = -k^2 \frac{b^2}{4\sigma}, \quad S_3 = -k^2 \left(1 - \frac{b^2}{4\sigma}\right), \\ \mu = k^2 \left(\frac{3b^2}{4\sigma} - 1\right), \end{aligned} \quad (154)$$

the stresses being negative (positive pressure) and the density positive.

The metric form (153) admits a 4-parameter group of motions¹; for it is unchanged by the transformation

$$\begin{aligned} x^1 &= \bar{x}^1 + A_1, & x^2 &= \bar{x}^2 e^{-kB} + A_2, \\ x^3 &= \bar{x}^3 + A_3, & x^4 &= \bar{x}^4 + B, \end{aligned} \quad (155)$$

where the A 's and B are arbitrary constants.

¹ Universes admitting a 3-parameter group of motions have been investigated by TAUB [1951] (cf. McVITTIE [1956, p. 75]); he has constructed some remarkable universes which are empty ($R_{ij} = 0$), but for which the Riemann tensor R_{ijkl} does not vanish.

To get isotropic pressure ($S_2 = S_3 = S_4 = -p$), we must restrict the constants in (153) by the equation

$$b^2 = 2ac, \quad (156)$$

and this gives

$$p = \mu = \frac{1}{2}k^2, \quad (157)$$

a rather disappointing result physically, since we would like to see p/μ small ¹.

Without the restriction (156), there appear to be four disposable constants in (153), but there are essentially only two. For if we apply the transformation

$$\begin{aligned} \sqrt{-a}x^1 &= k^{-1}\bar{x}^1, & \sqrt{-c}x^2 &= \frac{1}{\sqrt{2}}k^{-1}\bar{x}^2, \\ x^3 &= k^{-1}\bar{x}^3, & x^4 &= k^{-1}\bar{x}^4, \end{aligned} \quad (158)$$

we get

$$\begin{aligned} \Phi &= k^{-2}[-(dx^1)^2 - 2\lambda e^{\bar{x}^4}d\bar{x}^1d\bar{x}^2 - \frac{1}{2}e^{2\bar{x}^4}(d\bar{x}^2)^2 \\ &\quad + (d\bar{x}^3)^2 + (d\bar{x}^4)^2], \end{aligned} \quad (159)$$

where

$$\frac{2}{3} < \lambda^2 = \frac{b^2}{2ac} < 2. \quad (160)$$

If we now impose (156), we get $\lambda^2 = 1$, and (159) becomes (except for trivial differences in notation) the metric of GÖDEL [1949]; a number of interesting properties are listed and proved in his paper.

It has seemed worth while to make a systematic descent from the form (119) (which admits a 2-parameter group of motions) to Gödel's form, because in the course of this descent there emerge such formulae as (133) which are not too complicated to be of possible use in the construction of model universes.

§ 5. STATICAL UNIVERSES

A statical space-time has, by definition, a metric form

$$\Phi = g_{\alpha\beta}dx^\alpha dx^\beta + g_{44}(dx^4)^2, \quad (g_{44} < 0) \quad (161)$$

¹ Our standard practice has been to use the formula $\kappa T_{ij} = -G_{ij}$ to find the energy tensor corresponding to any given metric. But if we choose to use a cosmological constant Λ , then we write $\kappa T_{ij} = \Lambda g_{ij} - G_{ij}$, and so get a different energy tensor. If in the above model we choose $\Lambda = \frac{1}{2}k^2$, we get from (157) a fluid with density $\mu = \frac{1}{2}k^2$ and pressure $p = 0$.

with coefficients independent of x^4 . The geometry of space-time then involves only the geometry of a three-dimensional space with metric tensor $\bar{g}_{\alpha\beta} = g_{\alpha\beta}$ and a function g_{44} of the space-coordinates x^α . We shall indicate by a bar subtensors and other quantities pertaining to space. Thus, introducing a function V for convenience, we have

$$\begin{aligned} g_{\alpha\beta} &= \bar{g}_{\alpha\beta}, & g_{\alpha 4} &= 0, & g_{44} &= -V^2, \\ g^{\alpha\beta} &= \bar{g}^{\alpha\beta}, & g^{\alpha 4} &= 0, & g^{44} &= -V^{-2}, \\ [\alpha\beta, \gamma] &= [\bar{\alpha}\bar{\beta}, \bar{\gamma}], & [4\alpha, 4] &= -[44, \alpha] = -VV_{,\alpha}, \\ \Gamma_{\alpha\beta}^\gamma &= \bar{\Gamma}_{\alpha\beta}^\gamma, & \Gamma_{\alpha 4}^4 &= V^{-1}V_{,\alpha}, & \Gamma_{44}^\alpha &= \bar{g}^{\alpha\beta}VV_{,\beta} \end{aligned} \quad (162)$$

Throughout these calculations we find that the presence of one suffix 4, or of three of them, destroys a term. By I-(85) we find the Riemann tensor to be

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= \bar{R}_{\alpha\beta\gamma\delta}, & R_{\alpha\beta\gamma 4} &= 0, \\ R_{\alpha 44\delta} &= \frac{1}{2}g_{44}\|_{\alpha\delta} - \frac{1}{4}g^{44}g_{44,\alpha}g_{44,\delta} = -VV_{\| \alpha\delta}, \end{aligned} \quad (163)$$

where the double vertical stroke indicates covariant differentiation in space with respect to $\bar{g}_{\alpha\beta}$. It follows that the Ricci tensor is

$$\begin{aligned} R_{\alpha\beta} &= \bar{R}_{\alpha\beta} + \frac{1}{2}g^{44}(g_{44}\|_{\alpha\beta} - \frac{1}{2}g^{44}g_{44,\alpha}g_{44,\beta}) = \bar{R}_{\alpha\beta} + V^{-1}V_{\| \alpha\beta}, \\ R_{\alpha 4} &= 0, \\ R_{44} &= \frac{1}{2}\Delta_2 g_{44} - \frac{1}{4}g^{44}\Delta_1 g_{44} = -V\Delta_2 V, \end{aligned} \quad (164)$$

where the operators Δ_1 and Δ_2 are defined by

$$\Delta_1 \phi = \bar{g}^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta}, \quad \Delta_2 \phi = \bar{g}^{\alpha\beta} \phi_{\| \alpha\beta}. \quad (165)$$

Thus Δ_2 is the Laplace operator in curved space. We have then

$$\begin{aligned} R_\alpha^\alpha &= \bar{R} + V^{-1}\Delta_2 V, & R_4^4 &= V^{-1}\Delta_2 V, \\ R &= \bar{R} + 2V^{-1}\Delta_2 V. \end{aligned} \quad (166)$$

For the Einstein tensor we have the following formulae:

$$\begin{aligned} G_{\alpha\beta} &= R_{\alpha\beta} - \frac{1}{2}\bar{g}_{\alpha\beta}R, & G_{\alpha 4} &= 0, & G_{44} &= \frac{1}{2}V^2\bar{R}, \\ G_\alpha^\alpha &= R_\alpha^\alpha - \frac{3}{2}R = -\frac{1}{2}\bar{R} - 2V^{-1}\Delta_2 V, \\ G_4^4 &= R_4^4 - \frac{1}{2}R = -\frac{1}{2}\bar{R}, \\ G_4^4 - G_\alpha^\alpha &= 2V^{-1}\Delta_2 V. \end{aligned} \quad (167)$$

The most interesting thing here is the operator Δ_2 , for it enables us to connect a surface integral and a volume integral. Let v_3 be any portion of space, bounded by a closed surface v_2 , and let dv_3 and dv_2 denote their invariant elements of volume and area respectively. Then, by Green's theorem,

$$\oint_{v_2} V_{,\alpha} n^\alpha dv_2 = \int_{v_3} \Delta_2 V dv_3 = \frac{1}{2} \int_{v_3} (G_4^4 - G_\alpha^\alpha) V dv_3, \quad (168)$$

where n^α is the outward unit normal to v_2 .

So far we have done nothing but make calculations for the metric form (161). No field equations have been used. Let us now introduce them, writing $G_{ij} = -\kappa T_{ij}$, $\kappa = 8\pi$. It is at once evident from (168) that, if v_2 lies entirely in vacuo and can be reduced to a point without meeting matter, then

$$\oint_{v_2} V_{,\alpha} n^\alpha dv_2 = 0. \quad (169)$$

Hence this integral has the same value for any two surfaces which can be deformed into one another without meeting matter. In a statical universe we do not expect to find more than one body (cf. § 1), and for that one body there will be an exterior domain E and an interior domain I . Then (168) gives the *theorem of Gauss for a statical universe*¹:

$$\oint_{v_2} V_{,\alpha} n^\alpha dv_2 = 4\pi m, \quad (170)$$

where v_2 is any surface enclosing the body and m a constant defined by the body, viz.

$$m = - \int_{v_3} (T_4^4 - T_\alpha^\alpha) V dv_3, \quad (171)$$

the integral being taken through the body.

In the vacuum part of a static field, the field equations read

$$\bar{R}_{\alpha\beta} + V^{-1} V_{||\alpha\beta} = 0, \quad \Delta_2 V = 0. \quad (172)$$

We note that space has a vanishing curvature invariant \bar{R} .

We have already seen in § 1 how to handle a statical vacuum field with axial symmetry, a particular case of (161). Let us now make a

¹ Cf. WHITTAKER [1935]. In VII-§ 6 we already met the theorem of Gauss in the special case of spherical symmetry. The constant m occurring in the Schwarzschild metric VII-(145) is identical with the value given by (170); this is most easily seen from (179) below.

different simplification of the metric (161), taking

$$\Phi = U^2 dx^\alpha dx^\alpha - V^2 dt^2, \quad (173)$$

where U and V are independent of t ($= x^4$). Since the spatial metric is conformally flat, we may call this a *conformastat* metric. By direct calculation we obtain (for simplicity we denote partial derivatives by subscripts without commas)

$$\begin{aligned} R_{\alpha\beta\gamma\delta} = \bar{R}_{\alpha\beta\gamma\delta} = & U(\delta_{\alpha\delta}U_{\beta\gamma} + \delta_{\beta\gamma}U_{\alpha\delta} - \delta_{\alpha\gamma}U_{\beta\delta} - \delta_{\beta\delta}U_{\alpha\gamma}) \\ & - 2(\delta_{\alpha\delta}U_\beta U_\gamma + \delta_{\beta\gamma}U_\alpha U_\delta - \delta_{\alpha\gamma}U_\beta U_\delta - \delta_{\beta\delta}U_\alpha U_\gamma) \\ & + (\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\beta\delta}\delta_{\alpha\gamma})U_\sigma U_\sigma, \end{aligned} \quad (174)$$

$$R_{\alpha\beta\gamma 4} = 0,$$

$$\begin{aligned} R_{4\alpha\beta 4} = R_{\alpha 44\beta} = & -VV_{\alpha\beta} + VU^{-1}(U_\alpha V_\beta + U_\beta V_\alpha) \\ & - VU^{-1}\delta_{\alpha\beta}U_\sigma V_\sigma, \end{aligned}$$

and

$$\begin{aligned} R_{\alpha\beta} = & U^{-1}(U_{\alpha\beta} + \delta_{\alpha\beta}U_{\sigma\sigma}) - 2U^{-2}U_\alpha U_\beta \\ & + V^{-1}V_{\alpha\beta} - (UV)^{-1}(U_\alpha V_\beta + U_\beta V_\alpha) + (UV)^{-1}\delta_{\alpha\beta}U_\sigma V_\sigma \\ R_{\alpha 4} = & 0, \end{aligned} \quad (175)$$

$$R_{44} = -U^{-2}V(V_{\sigma\sigma} + U^{-1}U_\sigma V_\sigma),$$

$$R = 4U^{-3}(U_{\sigma\sigma} - \tfrac{1}{2}U^{-1}U_\sigma U_\sigma) + 2U^{-2}V^{-1}(V_{\sigma\sigma} + U^{-1}U_\sigma V_\sigma).$$

The above calculations hold for any conformastat metric. Let us try to satisfy the vacuum field equations $R_{ij} = 0$. The equations $R_{44} = 0$, $R = 0$ require that U and V satisfy the two equations

$$V_{\sigma\sigma} + U^{-1}U_\sigma V_\sigma = 0, \quad U_{\sigma\sigma} - \tfrac{1}{2}U^{-1}U_\sigma U_\sigma = 0, \quad (176)$$

the second of which is equivalent to

$$(\sqrt{U})_{\sigma\sigma} = 0, \quad (177)$$

so that \sqrt{U} must be a harmonic function with respect to the flat metric $dx^\alpha dx^\alpha$. Having chosen a harmonic function, the first of (176) gives V , but the determination of a vacuum conformastat field looks rather hopeless, since there are five more field equations to be satisfied.

However, at least one solution does exist, namely the exterior

Schwarzschild field as in VII-(145), for by the transformation

$$r = \rho \left(1 + \frac{m}{2\rho} \right)^2 \quad (178)$$

that metric form can be changed into the so-called *isotropic* form

$$\begin{aligned} \Phi &= U^2 dx^\alpha dx^\alpha - V^2 dt^2, \\ U &= (1 + \xi)^2, \quad V = \frac{1 - \xi}{1 + \xi}, \quad \xi = \frac{m}{2\rho}, \quad \rho^2 = x^\alpha x^\alpha. \end{aligned} \quad (179)$$

We check that \sqrt{U} is harmonic as in (177), and note that $(V - 1)/(V + 1)$ is also harmonic.

Returning to (175) and putting $UV = 1$, we note for future reference that for the form

$$\Phi = U^2 dx^\alpha dx^\alpha - U^{-2} dt^2 \quad (180)$$

we have

$$\begin{aligned} R_{\alpha\beta} &= \delta_{\alpha\beta} U^{-1} (U_{\sigma\sigma} - U^{-1} U_\sigma U_\sigma) + 2U^{-2} U_\alpha U_\beta, \\ R_{\alpha 4} &= 0, \\ R_{44} &= U^{-5} (U_{\sigma\sigma} - U^{-1} U_\sigma U_\sigma), \\ R &= 2U^{-3} U_{\sigma\sigma}. \end{aligned} \quad (181)$$

We recall that in v-§ 3 the Riemann, Ricci and Einstein tensors were calculated for another special conformastat metric, namely

$$\Phi = (1 + \phi) dx^\alpha dx^\alpha - (1 - \phi) dt^2. \quad (182)$$

For geometrical optics in a statical universe filled with a transparent medium, see XI-§ 4.

CHAPTER IX

GRAVITATIONAL WAVES

§ 1. PLANE GRAVITATIONAL WAVES

Before entering into a technical discussion, it is desirable to clarify the meaning of the term *gravitational wave*, and this is best done, not by formal definition, but by consideration of a fanciful experiment.

Suppose that a man, standing on the earth, holds in his hand a heavy club. At first the club hangs down towards the ground, but at a certain moment the man raises it quickly over his head. Any theory of gravitation recognizes that the club produces a gravitational field, however minute it may be, and that the action of the man changes that field, not only in his neighbourhood, but throughout the whole universe. According to Newtonian theory, the effect is instantaneously felt on the moon, on the sun and in every remote nebula. Since we are not concerned with Newtonian theory, we do not have to discuss the absurdity of this. As relativists, familiar with the idea that no causal effect can travel faster than light, and having learned, as in v-§ 7, that the locus of discontinuities of $g_{ij,km}$ is a null surface, we would guess that the change in the gravitational field of the moving club travels out into space with the speed of light. And we would call this moving disturbance a *gravitational wave*. Thus, on a very general basis, we must regard the physical existence of gravitational waves, so understood, as self-evident.

Confusion enters, however, through the fact that the word *wave* ¹ sometimes implies repetition and sometimes does not. In physics, this confusion is increased by the use of Fourier transforms, by which a disturbance which appears to be without repetition (a solitary wave) is resolved into periodic wave-trains with all frequencies ².

¹ The Oxford English Dictionary devotes nearly two pages to the noun *wave* and about as much to the verb.

² Since the technique of Fourier transforms is essentially a technique for dealing with *linear* differential equations, we are hardly likely to use it in connection with the *non-linear* equations of general relativity.

In speaking of waves, we shall not here demand that they should be repetitious. In v-§ 7 we have dealt with gravitational shock waves. We might call these shock waves *thin* to distinguish them from the *thick* gravitational waves ¹ which we shall now discuss.

Fig. 1 shows a thick gravitational wave in space-time. Two 3-spaces, Σ_1 and Σ_2 , divide space-time into three domains, I, II and III. There is no matter present anywhere in the universe (the man with the swinging club has been abstracted!), and everywhere we have

$$R_{ij} = 0. \quad (1)$$

In I and III there is no gravitational field, and so in those domains

$$R_{ijkl} = 0. \quad (2)$$

The domain II is the thick gravitational wave. In it, at least one of the components of the Riemann tensor is non-zero, and this we indicate by writing

$$R_{ijkl} \neq 0. \quad (3)$$

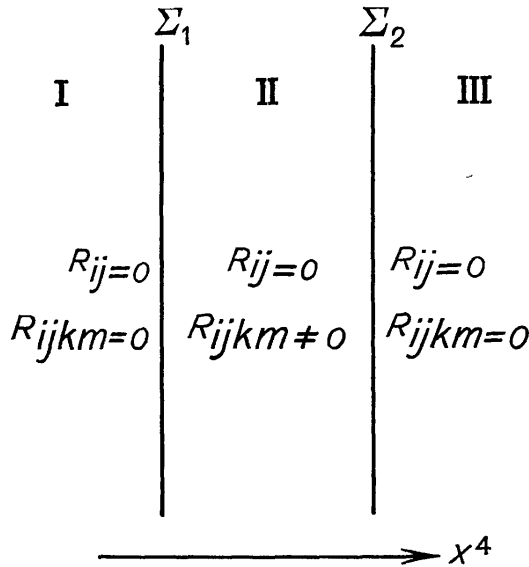


Fig. 1 – A thick gravitational wave in space-time

We recall that for admissible coordinates g_{ij} and $g_{ij,k}$ are continuous, but that there may be discontinuities in $g_{ij,km}$. The argument used below requires *three* systems of admissible coordinates ². One system covers II and the adjacent parts of I and III. Another covers I, not necessarily including the boundary Σ_1 . The third covers III, not necessarily including Σ_2 . In the overlaps, which are in I and in III, the transformations are C^3 , as required in I-§ 1.

On Σ_1 and Σ_2 we may have thin gravitational waves (shock waves) with discontinuities in $g_{ij,km}$, but this is not required. For the essential feature of the thick gravitational wave is the existence of a domain which is not flat, sandwiched between two flat domains. There is a

¹ Cf. BONDİ [1957a], BONDİ, PIRANI and ROBINSON [1959]; the second of these papers contains references of historical interest.

² If this offends formalists who would like to see one single metric form for all space-time, they should reflect on the ordinary spherical surface which cannot be adequately covered by a single coordinate system.

gravitational field in the inner domain, and none outside it.

Our point of departure for the discussion of thick gravitational waves is the metric form VIII-(119):

$$\Phi = g_{\alpha\beta} dx^\alpha dx^\beta + g_{AB} dx^A dx^B. \quad (4)$$

As in VIII-§ 4, Greek suffixes take the values 1, 2 and capital suffixes the values 3, 4. The coefficients $g_{\alpha\beta}$ are functions of (x^3, x^4) and g_{AB} are constants. However, we shall at once specialize this form by taking $g_{\alpha\beta}$ to be functions of x^4 only and setting

$$g_{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = g^{AB}, \quad (5)$$

so that (4) becomes

$$\Phi = g_{\alpha\beta} dx^\alpha dx^\beta + 2dx^3 dx^4. \quad (6)$$

Since

$$2dx^3 dx^4 = d\zeta^2 - d\tau^2, \quad (7)$$

$$\zeta = \frac{1}{\sqrt{2}}(x^3 + x^4), \quad \tau = \frac{1}{\sqrt{2}}(x^3 - x^4),$$

the form (6) has the required signature $+2$ provided $g_{\alpha\beta} dx^\alpha dx^\beta$ is positive-definite.

The formulae VIII-(123) are applicable, and, since $g^{44} = 0$, the surviving components of the Riemann tensor are

$$\begin{aligned} R_{1414} &= -\frac{1}{2}g_{11,44} + \frac{1}{4}g^{\rho\sigma}g_{\rho 1,4}g_{\sigma 1,4}, \\ R_{1424} &= -\frac{1}{2}g_{12,44} + \frac{1}{4}g^{\rho\sigma}g_{\rho 1,4}g_{\sigma 2,4}, \\ R_{2424} &= -\frac{1}{2}g_{22,44} + \frac{1}{4}g^{\rho\sigma}g_{\rho 2,4}g_{\sigma 2,4}. \end{aligned} \quad (8)$$

On calculating the Ricci tensor, we find (and this is most important) only *one* surviving component,

$$R_{44} = -g^{11}R_{1414} - 2g^{12}R_{1424} - g^{22}R_{2424}. \quad (9)$$

We now consider the construction of a thick plane gravitational wave with x^4 constant ¹ on Σ_1 and Σ_2 (Fig. 1). The problem is to ob-

¹ Since $g^{44} = 0$, Σ_1 and Σ_2 are null 3-spaces; in physical language, the gravitational wave is a plane wave, travelling in the ζ -direction [cf. (7)] with the speed of light ($d\zeta/d\tau = 1$).

tain g_{11} , g_{12} , g_{22} as functions of x^4 , of class C^1 in II and the adjacent parts of I and III, satisfying

$$R_{1414} = 0, \quad R_{1424} = 0, \quad R_{2424} = 0 \quad (10)$$

in those parts of I and III, and satisfying

$$R_{44} = 0 \quad (11)$$

in II, with violation of at least one of (10) ¹.

Before constructing a particular example of a thick gravitational wave (§ 3), let us survey the situation. In II, three functions are subject to only one equation, and it should be possible to select a wide variety of functions $g_{\alpha\beta}$ which violate at least one of (10). Having selected such functions, we have definite values of $g_{\alpha\beta}$ and $g_{\alpha\beta,4}$ on Σ_1 and Σ_2 . With these Cauchy data, (10) determine $g_{\alpha\beta}$ in the parts of I and III adjacent to Σ_1 and Σ_2 . Those parts will be flat, and we complete the construction by coordinate transformations in I and III to avoid formal singularities occurring in the Cauchy solutions. Apart from accidents, such as might arise from positive-definite character, this argument indicates the existence of a wide variety of thick gravitational waves.

Let us now simplify the metric (6) by taking

$$g_{11} = e^{2P}, \quad g_{12} = 0, \quad g_{22} = e^{2Q}, \quad (12)$$

P and Q being for the present arbitrary functions of x^4 . Then, denoting derivatives by primes, (8) gives only two survivors,

$$R_{1414} = -(P'' + P'^2)e^{2P}, \quad R_{2424} = -(Q'' + Q'^2)e^{2Q}, \quad (13)$$

and (9) gives

$$R_{44} = P'' + P'^2 + Q'' + Q'^2. \quad (14)$$

Thus to construct a gravitational wave with the metric form

$$\Phi = e^{2P}(dx^1)^2 + e^{2Q}(dx^2)^2 + 2dx^3dx^4, \quad (15)$$

in I and III we have to satisfy

$$P'' + P'^2 = 0, \quad Q'' + Q'^2 = 0, \quad (16)$$

so that

$$P = \log m(x^4 + \alpha), \quad Q = \log n(x^4 + \beta), \quad (17)$$

¹ If we did not violate at least one of (10), we would have completely flat space-time. and no wave at all.

where α , β , m , n are constants (different in I and III), and in II we have to satisfy

$$P'' + P'^2 + Q'' + Q'^2 = 0, \quad (18)$$

taking care *not* to satisfy both of (16). Since this last equation may be written

$$P'' + Q'' = -P'^2 - Q'^2 < 0, \quad (19)$$

an interesting fact emerges. If we plot (P', Q') as a point in a plane (Fig. 2), and draw the lines $P' + Q' = \text{const.}$, the representative point moves across these lines in the sense indicated. Thus no solution of (18) can form a closed curve in the plane; this means that a periodic field is impossible.

A particular gravitational wave will be given in § 3.

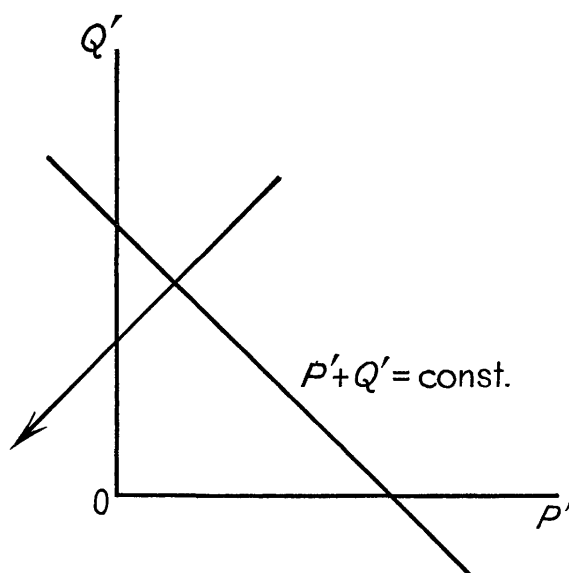


Fig. 2 – The tendency inside a gravitational wave

§ 2. THE WORLD-FUNCTION FOR A PLANE GRAVITATIONAL WAVE AND QUASI-CARTESIAN COORDINATES

The following work is useful in transforming to regular form the metrics in the flat domains I and III of a plane gravitational wave. But it has a wider scope. We shall evaluate the world-function Ω for the metric form (15), without at first imposing any restrictions of the functions $P(x^4)$ and $Q(x^4)$.

To find the geodesics of (15), we write the Lagrangian

$$F = \frac{1}{2}[e^{2P}(Dx^1)^2 + e^{2Q}(Dx^2)^2 + 2Dx^3Dx^4], \quad (20)$$

where $D = d/ds$, and, since all coordinates are ignorable except x^4 , we get the three first integrals

$$e^{2P}dx^1 = \alpha_1 ds, \quad e^{2Q}dx^2 = \alpha_2 ds, \quad dx^4 = \beta^{-1}ds, \quad (21)$$

where α_1 , α_2 and β are constants. We have also

$$e^{2P}(dx^1)^2 + e^{2Q}(dx^2)^2 + 2dx^3dx^4 = \varepsilon ds^2, \quad (22)$$

where $\varepsilon (= \pm 1)$ is the indicator of the geodesic, supposed timelike or

spacelike. Hence, in terms of x^4 , we get

$$\begin{aligned} dx^1 &= \beta\alpha_1 e^{-2P} dx^4, & dx^2 &= \beta\alpha_2 e^{-2Q} dx^4, & ds &= \beta dx^4, \\ dx^3 &= \frac{1}{2}\beta^2 dx^4 (\varepsilon - \alpha_1^2 e^{-2P} - \alpha_2^2 e^{-2Q}). \end{aligned} \quad (23)$$

Consider the geodesic joining the points $A(x^i)$ and $A'(x^{i'})$. Writing

$$\xi^i = x^i - x^{i'}, \quad I_1 = \int_{x^{4'}}^{x^4} e^{-2P} dx^4, \quad I_2 = \int_{x^{4'}}^{x^4} e^{-2Q} dx^4, \quad (24)$$

we get

$$\begin{aligned} \xi^1 &= \beta\alpha_1 I_1, & \xi^2 &= \beta\alpha_2 I_2, \\ \xi^3 &= \frac{1}{2}\beta^2 \varepsilon \xi^4 - \frac{1}{2}\beta^2 \alpha_1^2 I_1 - \frac{1}{2}\beta^2 \alpha_2^2 I_2, & s &= \beta \xi^4. \end{aligned} \quad (25)$$

Hence

$$\xi^3 = \frac{1}{2}\varepsilon\beta^2\xi^4 - \frac{1}{2} \frac{(\xi^1)^2}{I_1} - \frac{1}{2} \frac{(\xi^2)^2}{I_2}, \quad (26)$$

and so

$$\frac{1}{2}\varepsilon\beta^2 = \frac{1}{\xi^4} \left[\xi^3 + \frac{1}{2} \frac{(\xi^1)^2}{I_1} + \frac{1}{2} \frac{(\xi^2)^2}{I_2} \right]. \quad (27)$$

Thus the world-function is

$$\Omega(AA') = \frac{1}{2}\varepsilon s^2 = \frac{1}{2}\varepsilon\beta^2(\xi^4)^2 = \xi^4 \left[\xi^3 + \frac{1}{2} \frac{(\xi^1)^2}{I_1} + \frac{1}{2} \frac{(\xi^2)^2}{I_2} \right]. \quad (28)$$

We shall now pass from the coordinates x^i to quasi-Cartesian (QC)

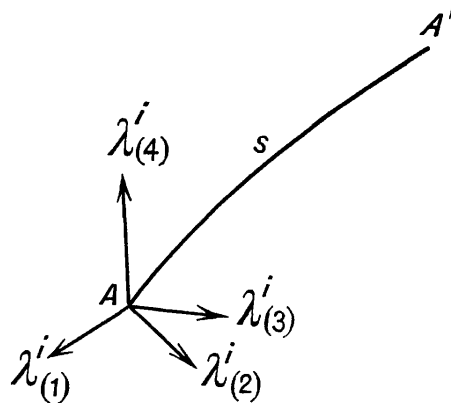


Fig. 3 – Construction for quasi-Cartesian coordinates of A' relative to a vector base at A

coordinates $X_{(a)}$ as in II–(150). At A (Fig. 3) choose an orthonormal tetrad $\lambda^i_{(a)}$ with $\lambda^i_{(4)}$ timelike. Then the QC of A' relative to A with this

vector base are

$$X_{(a)} = -\Omega_i(AA')\lambda_{(a)}^i, \quad (29)$$

where Ω_i is the partial derivative with respect to x^i at A . By (28) we have ¹

$$\begin{aligned} \Omega_1 &= \frac{\xi^1 \xi^4}{I_1}, & \Omega_2 &= \frac{\xi^2 \xi^4}{I_2}, & \Omega_3 &= \xi^4, \\ \Omega_4 &= \xi^3 + \frac{1}{2} \frac{(\xi^1)^2}{I_1} + \frac{1}{2} \frac{(\xi^2)^2}{I_2} - \frac{1}{2} \frac{\xi^4 (\xi^1)^2 e^{-2P}}{I_1^2} \\ &\quad - \frac{1}{2} \frac{\xi^4 (\xi^2)^2 e^{-2Q}}{I_2^2}, \end{aligned} \quad (30)$$

with P and Q evaluated at A . For the orthonormal tetrad, let us take

$$\begin{aligned} \lambda_{(1)}^i &: (e^{-P}, 0, 0, 0), \\ \lambda_{(2)}^i &: (0, e^{-Q}, 0, 0), \\ \lambda_{(3)}^i &: \left(0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \\ \lambda_{(4)}^i &: \left(0, 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right). \end{aligned} \quad (31)$$

Then (29) gives as QC for A' (the coordinates $x^{i'}$ of A' are concealed in ξ^i and in I_1, I_2)

$$\begin{aligned} X_{(1)} &= -e^{-P} \frac{\xi^1 \xi^4}{I_1}, & X_{(2)} &= -e^{-Q} \frac{\xi^2 \xi^4}{I_2}, \\ X_{(3)} &= \frac{1}{\sqrt{2}} \left[-\xi^4 - \xi^3 - \frac{1}{2} \frac{(\xi^1)^2}{I_1} - \frac{1}{2} \frac{(\xi^2)^2}{I_2} \right. \\ &\quad \left. + \frac{1}{2} \frac{\xi^4 (\xi^1)^2 e^{-2P}}{I_1^2} + \frac{1}{2} \frac{\xi^4 (\xi^2)^2 e^{-2Q}}{I_2^2} \right], \\ X_{(4)} &= \frac{1}{\sqrt{2}} \left[\xi^4 - \xi^3 - \frac{1}{2} \frac{(\xi^1)^2}{I_1} - \frac{1}{2} \frac{(\xi^2)^2}{I_2} \right. \\ &\quad \left. + \frac{1}{2} \frac{\xi^4 (\xi^1)^2 e^{-2P}}{I_1^2} + \frac{1}{2} \frac{\xi^4 (\xi^2)^2 e^{-2Q}}{I_2^2} \right]. \end{aligned} \quad (32)$$

¹ Remembering that $g^{34} = 1$, we easily check that the basic equation $2\Omega = g^{ij}\Omega_i\Omega_j$ is satisfied; cf. II-(20).

As a check on these formulae, we easily verify that

$$X_{(1)}^2 + X_{(2)}^2 + X_{(3)}^2 - X_{(4)}^2 = 2\Omega(AA'). \quad (33)$$

We recall that all the preceding formulae in this section apply to the metric form

$$\Phi = e^{2P}(dx^1)^2 + e^{2Q}(dx^2)^2 + 2dx^3dx^4, \quad (34)$$

without restriction on the functions $P(x^4)$, $Q(x^4)$. For gravitational waves, P and Q are finite in the domain II (Fig. 1) and in the neighbouring parts of I and III, but they have formal singularities elsewhere in I and III. However, I and III are flat, and we can transform their metrics separately to Minkowskian form by substituting from (17) in (32). This gives

$$\begin{aligned} X_{(1)} &= -m\xi^1(x^{4'} + \alpha), & X_{(2)} &= -n\xi^2(x^{4'} + \beta), \\ X_{(3)} - X_{(4)} &= -\sqrt{2}\xi^4, \end{aligned} \quad (35)$$

$$X_{(3)} + X_{(4)} = -\sqrt{2}[\xi^3 + \frac{1}{2}m^2(\xi^1)^2(x^{4'} + \alpha) + \frac{1}{2}n^2(\xi^2)^2(x^{4'} + \beta)].$$

In these formulae ξ^i is as in (24). The point x^i is any point in the domain of regularity of (17), and (35) is a transformation $(x^{i'}) \rightarrow (X_{(a)})$. It is easy to verify that

$$\begin{aligned} dX_{(1)}^2 + dX_{(2)}^2 + dX_{(3)}^2 - dX_{(4)}^2 = \\ e^{2P'}(dx^{1'})^2 + e^{2Q'}(dx^{2'})^2 + 2dx^{3'}dx^{4'}, \end{aligned} \quad (36)$$

the primes on P' and Q' indicating evaluation at $x^{i'}$.

§ 3. A PARTICULAR PLANE GRAVITATIONAL WAVE AND REMARKS ON CYLINDRICAL AND SPHERICAL WAVES

We shall now construct a particular thick plane gravitational wave, in order to show that the plan described in § 1 can actually be carried out. No special merit, except simplicity, attaches to this example.

Take Σ_1 and Σ_2 to be $x = -a$ and $x = a$, respectively (we write x for x^4). Then the domain II is $-a < x < a$, and in it we shall satisfy (18) by making P and Q satisfy

$$P'' + P'^2 = -k^{-2}, \quad Q'' + Q'^2 = k^{-2}, \quad k = \frac{4a}{\pi}. \quad (37)$$

As particular solutions we select

$$P = \log \cos (x/k), \quad Q = x/k, \quad (38)$$

so that

$$\begin{aligned} P' &= -k^{-1} \tan(x/k), & Q' &= k^{-1}, \\ P'' &= -k^{-2} \sec^2(x/k), & Q'' &= 0. \end{aligned} \quad (39)$$

This gives the end-values

$$\begin{aligned} \Sigma_1 (x = -a): \quad P &= \log \frac{1}{\sqrt{2}}, & Q &= -a/k, \\ P' &= k^{-1}, & Q' &= k^{-1}, \\ P'' &= -2k^{-2}, & Q'' &= 0; \end{aligned} \quad (40)$$

$$\begin{aligned} \Sigma_2 (x = a): \quad P &= \log \frac{1}{\sqrt{2}}, & Q &= a/k, \\ P' &= -k^{-1}, & Q' &= k^{-1}, \\ P'' &= -2k^{-2}, & Q'' &= 0. \end{aligned} \quad (41)$$

We have now to assign to the domains I and III functions P and Q of the form (17), with continuity of these functions and their first derivatives across Σ_1 and Σ_2 . We get

$$\begin{aligned} \text{In I: } e^P &= \frac{1}{k\sqrt{2}}(x + a + k), & e^Q &= e^{-\frac{1}{2}\pi} \cdot k^{-1}(x + a + k), \\ \text{In III: } e^P &= \frac{1}{k\sqrt{2}}(a + k - x), & e^Q &= e^{\frac{1}{2}\pi} \cdot k^{-1}(x - a + k). \end{aligned} \quad (42)$$

We note that formal singularities occur at

$$x = -a - k \text{ in I, } x = a + k \text{ in III.} \quad (43)$$

Thus, restoring the symbol x^4 , we have a gravitational wave with the following metrics:

$$\begin{aligned} \text{In part of I } (-a - k < x^4 \leq -a): \\ &= k^{-2}(x^4 + a + k)^2 [\tfrac{1}{2}(dx^1)^2 + e^{-\frac{1}{2}\pi}(dx^2)^2] + 2dx^3dx^4, \\ \text{In II } (-a \leq x^4 \leq a): \\ &= \cos^2(x^4/k)(dx^1)^2 + e^{2x^4/k}(dx^2)^2 + 2dx^3dx^4, \\ \text{In part of III } (a \leq x^4 < a + k): \\ &= \tfrac{1}{2}k^{-2}(x^4 - a - k)^2(dx^1)^2 + e^{\frac{1}{2}\pi} \cdot k^{-2}(x^4 - a + k)^2(dx^2)^2 + 2dx^3dx^4. \end{aligned} \quad (44)$$

As for the remaining infinite parts of the flat domains I and III, we can obtain as in (35) new coordinates for which the metric takes the Minskowskian form; in applying (35) we are to take a point x^i in the domains of I and III indicated in (44).

Although mathematically impeccable, the gravitational waves which we have been considering are not physically realistic. In electromagnetism, a travelling layer of disturbance sandwiched between two undisturbed regions is a reasonable idealization of reality, because we can project a pulse into undisturbed space, with no disturbance after it has passed. But while a man can swing a massive club, he cannot create the matter in it. He can change gravitational fields, but he cannot create them out of nothing. We should prefer to see, in our model, a wave of disturbance passing through a field already existent.

This difficulty is overcome in the case of cylindrical waves¹. For the metric form

$$\Phi = e^{2\gamma-2\psi}(dr^2 - dt^2) + r^2e^{-2\psi}d\phi^2 + e^{2\psi}dz^2, \quad (45)$$

with γ and ψ functions of r and t , the equations $R_{ij} = 0$ reduce to

$$\psi_{rr} + \frac{1}{r}\psi_r - \psi_{tt} = 0, \quad (46)$$

$$\gamma_r = r(\psi_r^2 + \psi_t^2), \quad \gamma_t = 2r\psi_r\psi_t. \quad (47)$$

Here (46) is the ordinary wave equation in a plane, and it is the condition of integrability of (47). On account of its linearity, we can superimpose solutions of (46) (a statical basic field and a time-dependent disturbance) and then get γ from (47) by quadrature.

But a cylindrical wave is not quite realistic enough. Our intuition tells us that the field of the swinging club must, at great distances, display spherical symmetry. But some lack of symmetry, some polarization, must be present, since a field with perfect spherical symmetry must, by Birkhoff's theorem (VII-§ 4), be static, in the sense that it admits a group of motions along timelike lines.

In seeking a general understanding of gravitational waves, due to a swinging club or to a vast catastrophe of astronomical scale, it is well to recognize that we are not concerned with the solution of well-

¹ EINSTEIN and ROSEN [1937], ROSEN [1954], MARDER [1958a, b], [1959], BONNOR [1957b]. There is a close formal connection between statical fields with axial symmetry and cylindrical waves. The formulae VIII-(18) and VIII-(24) become (45) and (46) if we make the substitution $(z, t) \rightarrow (it, iz)$ in the former.

formulated mathematical problems, but rather with classes of fields satisfying certain conditions. To fix our ideas, let us think of space-time as a Euclidean 4-space with coordinates (r, θ, ϕ, t) and a metric tensor g_{ij} imposed on that background. We divide space-time into an interior I ($r < a$) and an exterior E ($r > a$). In I there are some moving masses and E is empty, so that the equations $R_{ij} = 0$ are to be satisfied, and we should add the condition of flatness at infinity, which means that $R_{ijkl} \rightarrow 0$ as $r \rightarrow \infty$.

The study of gravitational waves does not involve essentially an investigation of I . On account of the prevalence of linear theories in physics, one is tempted to regard the field in E as 'due to' 'sources' in I , but this idea is deceptive in a non-linear theory. It is better to concentrate attention on E . Any solution of $R_{ij} = 0$ in E would be worthy of respect if it reflected our intuitive idea of what gravitational waves should be. But no mere guess is likely to succeed in such a complicated situation.

In default of exact solutions in E , we may fall back on approximations. Here we must be cautious. Mathematically, we have no assurance that any suitable solutions exist; but, physically, we do. So we set up some definite system of approximation, based probably on powers of a small parameter. Cutting off the approximation at some step, we may claim that we have a good approximation to some exact solution, which (physically, but not mathematically) we have reason to believe exists.

But such a claim is too vague to argue about, one way or the other. One fact is certain: the approximate g_{ij} do not satisfy $R_{ij} = 0$ in E . We may then fall back on the fact that any g_{ij} corresponds to some distribution of matter, perhaps pathological, and we may explore that distribution by examining, as in VIII-(116), the eigenvalues of G_{ij} . It may then be asserted that the approximate g_{ij} give some universe, and the practical physicist will accept as vacuum a space-time in which stress and density are small enough in comparison with standard quantities of the same dimensions. We may call this the *stress-density test*.

No linear approximation is likely to satisfy this test. Approximations pushed to higher orders are more hopeful, such as those of BONNOR [1959b], based on retarded potential formulae for a pair of oscillating masses. But in such work the formulae become so complicated that it is difficult to apply the stress-density test.

CHAPTER X

ELECTROMAGNETISM

§ 1. MAXWELL'S EQUATIONS AND THE ELECTROMAGNETIC ENERGY TENSOR

It is well known that, on the atomic scale, electromagnetic attractions and repulsions far exceed gravitational attractions, whereas, on the astronomical scale, it is the other way round, because celestial bodies are electrically neutral, or nearly so, so that the electrical effects cancel out. There is then some reason for keeping theories of gravitation and electromagnetism apart.

But we cannot do that. All electromagnetic phenomena are not small-scale, and we need a theory to enable us to follow radiation passing from a star to the earth through curved space-time. This suggests that we set electromagnetism in a space-time whose properties are determined by the masses in it, which amounts to neglecting the gravitational fields (if any) arising from electromagnetic energy¹. On the other hand, some have felt that gravitation and electromagnetism should be tied together in a very deep way in a single unified theory competent to deal with all physics from the atomic to the cosmic scale².

These hopes have not been fulfilled, and it seems reasonable to treat electromagnetism within the framework of general relativity as already developed in this book. But, even though the gravitational fields arising from electromagnetism are in fact very small, we shall make a unification of electromagnetism and gravitation at least to the extent of allowing the electromagnetic field to influence the geometry of space-time.

We shall consider only electromagnetic fields in vacuo or in an

¹ Cf. WHITTAKER [1927b], [1928a, b, c].

² Cf. WEYL [1918a], [1920], [1923a], EINSTEIN [1925b], [1955], SCHRÖDINGER [1947], [1948], [1950], and other references in Bibliography, particularly TONNELAT [1955] and HLAVATÝ [1957].

incoherent fluid composed of particles all charged with electricity of the same sign ¹. Then there are the following functions of the space-time coordinates x^i :

- g_{ij} = symmetric metric tensor,
- F_{ij} = skew-symmetric electromagnetic tensor ($F_{ij} = -F_{ji}$),
- J^i = 4-current,
- u^i = 4-velocity of charge, a unit vector ($u_i u^i = -1$),
- ρ = electrical proper density,
- μ = mass-density.

These quantities are not independent, for we have

$$J^i = \rho u^i. \quad (1)$$

Other differential relations will be introduced below. We can pass from a charged fluid to an uncharged fluid by putting $J^i = 0$, $\rho = 0$, and finally to a vacuum by putting $\mu = 0$.

Let $\lambda_{(a)}^i$ be an orthonormal tetrad with $\lambda_{(4)}^i$ timelike and future-pointing. From F_{ij} we can form the invariants

$$F_{(ab)} = F_{ij} \lambda_{(a)}^i \lambda_{(b)}^j = -F_{(ba)}. \quad (2)$$

Writing

$$\begin{aligned} F_{(14)} &= E_1, & F_{(24)} &= E_2, & F_{(34)} &= E_3, \\ F_{(23)} &= H_1, & F_{(31)} &= H_2, & F_{(12)} &= H_3, \end{aligned} \quad (3)$$

we introduce the familiar language of physics by calling ² E_α the *electric 3-vector* and H_α the *magnetic 3-vector*. Likewise we form the invariants

$$J_{(a)} = J_i \lambda_{(a)}^i; \quad (4)$$

we call $J_{(a)}$ the 3-current and $J_{(4)}$ the *electrical relative density* (not to be confused with the proper density ρ , which is also an invariant but more basic, since it does not depend on the choice of tetrad).

The permutation tensor 1-(114) is useful in electromagnetism. Writing

$$q = \sqrt{-g}, \quad (5)$$

¹ For more general situations, see PHAM MAU QUAN [1955a, b], [1956a, b], [1957a, b], [1958a, b].

² We revert to the usual convention of this book: Greek suffixes have the range 1, 2, 3.

we have the following formulae (note the minus signs!):

$$\begin{aligned}\eta^{ijkm} &= q^{-1}\varepsilon_{ijkm}, \quad \eta_{ijkm} = -q\varepsilon_{ijkm}, \\ \eta^{ijkm}\eta_{iabc} &= -\varepsilon_{ijkm}\varepsilon_{iabc} = -\delta_{abc}^{jkm}, \\ \eta^{ijkm}\eta_{ijbc} &= -2(\delta_b^k\delta_c^m - \delta_c^k\delta_b^m) = -2\delta_{bc}^{km}, \\ \eta^{ijkm}\eta_{ijkc} &= -6\delta_c^m.\end{aligned}\tag{6}$$

We already met the six-index generalized Kronecker delta in I-(122); the four-index delta is similarly defined. It is important to remember that the covariant derivative of the permutation tensor vanishes.

The *dual electromagnetic tensor* is defined by either of the following equivalent formulae ¹:

$$F^{*ij} = \frac{1}{2}\eta^{ijkm}F_{km}, \quad F_{ij}^* = \frac{1}{2}\eta_{ijkm}F^{km}.\tag{7}$$

Multiplying by η_{ijab} , η^{ijab} respectively, using (6), and changing the suffixes, we get

$$F^{ij} = -\frac{1}{2}\eta^{ijkm}F_{km}^*, \quad F_{ij} = -\frac{1}{2}\eta_{ijkm}F^{*km}.\tag{8}$$

We now accept *Maxwell's equations* in the form ²

$$F^{ij}_{|j} = J^i, \quad F_{ij,k} + F_{jk,i} + F_{ki,j} = 0.\tag{9}$$

It is of interest that the second of these equations is tensorial although the derivatives are partial. They may be replaced by covariant derivatives, however, and it is easy to see that Maxwell's equations may also be exhibited in the form ³

$$F^{ij}_{|j} = J^i, \quad F^{*ij}_{|j} = 0.\tag{10}$$

For any skew-symmetric tensor X^{ij} and for any vector Y^i we have, by I-(8), I-(10), I-(12),

$$\begin{aligned}X^{ij}_{|j} &= q^{-1}(qX^{ij})_{,j}, \quad Y^i_{|i} = q^{-1}(qY^i)_{,i}, \\ X^{ij}_{|ij} &= (X^{ij}_{|i})_{,j} = q^{-1}(qX^{ij}_{|i})_{,j} = q^{-1}(qX^{ij})_{,ij} = 0.\end{aligned}\tag{11}$$

Hence Maxwell's equations may be exhibited in yet a third form,

$$(qF^{ij})_{,j} = qJ^i, \quad (qF^{*ij})_{,j} = 0,\tag{12}$$

and the 4-current satisfies an *equation of conservation* which may be

¹ The star is used in a different sense in v-(63), but there should be no risk of confusion.

² No factor 4π appears in front of J^i if we use rational units of charge.

³ For details of the transformation, see § 3 below.

written in either of the following forms ¹:

$$J_{|i}^i = 0, \quad (qJ^i)_{,i} = 0. \quad (13)$$

By virtue of the second equation in (9), there exists a vector called the 4-*potential* ϕ_i such that

$$F_{ij} = \phi_{j,i} - \phi_{i,j} = \phi_{j|i} - \phi_{i|j}. \quad (14)$$

Substitution in the first of (9) gives

$$\square\phi_i - g^{ab}\phi_{a|ib} + J_i = 0, \quad (15)$$

where \square is the *generalized d'Alembertian*,

$$\square\phi_i = g^{ab}\phi_{i|ab}. \quad (16)$$

By 1-(94) we have the identities

$$\begin{aligned} \phi_{a|ib} - \phi_{a|bi} &= R_{.aib}^j \phi_j, \\ g^{ab}\phi_{a|ib} - (g^{ab}\phi_{a|b})_{|i} &= -R_{ij}\phi^j. \end{aligned} \quad (17)$$

If we impose on ϕ_i the normalizing condition ²

$$g^{ab}\phi_{a|b} = 0, \quad (18)$$

(17) gives

$$g^{ab}\phi_{a|ib} = -R_{ij}\phi^j, \quad (19)$$

and (15) becomes

$$\square\phi_i + R_{ij}\phi^j + J_i = 0. \quad (20)$$

We thus reduce Maxwell's equations to the five equations contained in (18) and (20), but of these only four are independent.

In vacuo we have $J_i = 0$ and, if we neglect the gravitational effect of the electromagnetic field, $R_{ij} = 0$. Then (20) reduces to the generalized wave equation,

$$\square\phi_i = 0. \quad (21)$$

To link electromagnetism with gravitation, we now assign to the charged fluid an energy tensor ³

$$T^{ij} = \mu u^i u^j + E^{ij}, \quad (22)$$

¹ qF^{ij} , qF^{*ij} and qJ^i are *tensor densities* or *relative tensors of weight 1*; cf. SYNGE and SCHILD [1956, p. 240].

² Combining (14) and (18), the determination of ϕ_i may be treated as a Cauchy problem.

³ The 4-velocity V^i , defined [cf. IV-(75)] as the unit timelike eigenvector of the energy tensor, would represent in the present case a synthesis of charge and field, and must not be confused with u^i , which refers to charge alone, and is not an eigenvector of the energy tensor (22).

where

$$E^{ij} = g_{ab}F^{ai}F^{bj} - \frac{1}{4}g^{ij}F_{ab}F^{ab}. \quad (23)$$

This energy tensor consists of two parts, one due to the matter which carries the charge and the other to the field alone. With regard to the latter, we note that

$$\begin{aligned} g_{ij}E^{ij} &= g_{ij}g_{ab}F^{ai}F^{bj} - F_{ab}F^{ab} \\ &= g_{ij}g_{ab}F^{ai}F^{bj} - g_{ia}g_{jb}F^{ij}F^{ab}. \end{aligned} \quad (24)$$

Interchanging the dummies a, j in the last term, we see that it cancels with the first term. Thus

$$E_i^i = 0; \quad (25)$$

the mixed energy tensor of the electromagnetic field has zero trace.

We now write down the usual field equations

$$G_{ij} = -\kappa T_{ij}, \quad \kappa = 8\pi, \quad (26)$$

and (except for coordinate conditions) this completes the system of equations for a charged fluid. We shall presently examine the Cauchy problem for this system, but let us find the *equations of motion* arising from the application to (26) of the identity

$$G^{ij}{}_{|j} = 0. \quad (27)$$

From (23) we have

$$E^{ij}{}_{|j} = g_{ab}F^{ai}{}_{|j}F^{bj} + g_{ab}F^{ai}F^{bj}{}_{|j} - \frac{1}{2}g^{ij}F_{ab|j}F^{ab}, \quad (28)$$

or, by the first of (9) and some play with indices,

$$\begin{aligned} E^{ij}{}_{|j} + F^{ij}J_j &= g_{ab}F^{ai}{}_{|j}F^{bj} - \frac{1}{2}g^{ij}F_{ab|j}F^{ab} \\ &= g^{ij}F_{bj|a}F^{ba} - \frac{1}{2}g^{ij}F_{ab|j}F^{ab} \\ &= \frac{1}{2}g^{ij}F^{ab}(F_{aj|b} - F_{bj|a} - F_{ab|j}) \\ &= \frac{1}{2}g^{ij}F^{ab}(F_{aj|b} + F_{jb|a} + F_{ba|j}). \end{aligned} \quad (29)$$

By the second of (9) this vanishes, and so, having used all Maxwell's equations, we have

$$E^{ij}{}_{|j} = -F^{ij}J_j. \quad (30)$$

This is the reason why (23) is a suitable expression for the energy tensor of the field: in vacuo its divergence vanishes.

From (27) we now obtain

$$(\mu u^i u^j)_{|j} = F^{ij} J_j, \quad (31)$$

or

$$u^i(\mu u^j)_{|j} + \mu u^i_{|j} u^j = F^{ij} J_j. \quad (32)$$

Multiply by u_i . The second term vanishes since $u_i u^i = -1$, and the right hand side vanishes from the skew-symmetry of F^{ij} and the fact that by (1) u^i has the direction of J^i . Thus

$$(\mu u^j)_{|j} = 0, \quad (33)$$

which is an equation of *conservation of mass*. Then (32) reduces to

$$\mu u^i_{|j} u^j = F^{ij} J_j, \quad (34)$$

or, with absolute differentiation along the u -line,

$$\mu \frac{\delta u^i}{\delta s} = F^{ij} J_j, \quad (35)$$

or

$$\mu \frac{\delta u^i}{\delta s} = \rho F^{ij} u_j. \quad (36)$$

Now (13) may be written

$$(\rho u^i)_{|i} = 0, \quad (37)$$

an equation of *conservation of charge*. Combined with (33), this tells us that if we take a thin tube of u -lines with normal section σ , then the total mass $m = \mu\sigma$ and the total charge $e = \rho\sigma$ are conserved as we go along the tube. Thus, with m and e constants for the tube, we may write (36) in the form

$$m \frac{\delta u^i}{\delta s} = e F^{ij} u_j. \quad (38)$$

We are dealing with continuous field theory in which a charged point-particle has no meaning. But, just as we accepted the geodesic hypothesis for an uncharged test-particle, we may accept (38) as the equation of motion of a charged test-particle of mass m , charge e and 4-velocity u^i , moving in a given field of gravitation (g_{ij}) and electromagnetism (F_{ij}), which field the particle itself does not influence. We have here in fact the natural generalization to curved space-time of the Heaviside-Lorentz law of ponderomotive force ¹.

¹ Cf. SYNGE [1956a, p. 394].

§ 2. THE CAUCHY PROBLEM FOR AN INCOHERENT CHARGED FLUID

Let us introduce the notation

$$Z_{ij} = G_{ij} + \kappa T_{ij}, \quad \nu = \mu/\rho^2, \quad (39)$$

with

$$\begin{aligned} T_{ij} &= \nu J_i J_j + E_{ij}, \\ E_{ij} &= g^{ab} F_{ai} F_{bj} - \frac{1}{4} g_{ij} F_{ab} F^{ab}. \end{aligned} \quad (40)$$

This is the energy tensor (22), written a little differently. Then the set of field equations for an incoherent charged fluid read

$$Z_{ij} = 0, \quad F^{ij}{}_{|j} = J^i, \quad F^{*ij}{}_{|j} = 0, \quad C_i(g) = 0, \quad (41)$$

the last being four coordinate conditions.

We see 21 unknowns,

$$g_{ij}, \quad F_{ij}, \quad J_i, \quad \nu; \quad (42)$$

there appear to be $10 + 4 + 4 + 4 = 22$ equations in (41), but only 21 are independent on account of the identity [cf. (11)]

$$F^{*ij}{}_{|ij} = 0. \quad (43)$$

Thus, on the mere basis of counting, we appear to have in (41) a determinate system; when the quantities (42) have been found, the other quantities are given by

$$\rho = (-J_i J^i)^{\frac{1}{2}}, \quad u_i = \rho^{-1} J_i, \quad \mu = \nu \rho^2. \quad (44)$$

We shall now examine the Cauchy problem¹ for the system (41). Taking skew-Gaussian coordinates x^i relative to a 3-space $x^4 = 0$, we have the coordinate conditions

$$g_{\alpha 4, 4} = 0, \quad g_{44} = \pm 1. \quad (45)$$

On $x^4 = 0$ we assign as Cauchy data (CD) the values of

$$g_{\alpha\beta}, \quad g_{\alpha\beta, 4}, \quad g_{\alpha 4}, \quad F_{ij}, \quad J_i, \quad \nu, \quad (46)$$

subject to certain conditions to be given later, and we investigate the

¹ Cf. LICHNEROWICZ [1955a, p. 55]. For other work on the Cauchy problem, see FOURÈS-BRUHAT [1948b], [1950], [1952], [1955], [1956], PHAM MAU QUAN [1953b], [1955b].

algebraic problem of solving the equations (41) for

$$g_{\alpha\beta,44}, \quad F_{ij,4}, \quad J_{i,4}, \quad v_{,4}. \quad (47)$$

The equations $Z_{ij} = 0$ are equivalent to

$$R_{ij} = -\kappa(T_{ij} - \tfrac{1}{2}g_{ij}T_k^k), \quad (48)$$

where the right hand side is CD. Now the quantities $g_{\alpha\beta,44}$ occur only in these equations, and, as at v-(171), the solution is not unique if $g^{44} = 0$. Thus null surfaces are shock waves (characteristics). This holds in vacuo ($J_i = 0$), and therefore *electromagnetic shock waves in vacuo are null surfaces*. In particular, in so far as they may be regarded as shock waves, waves of light are null surfaces. Further, the bicharacteristics are null geodesics [cf. v-§ 7], and this gives us confidence in the geodesic hypothesis for photons (photons, like point-particles, lie outside field theory).

Assuming then that $x^4 = 0$ is not a null surface ($g^{44} \neq 0$), Lemma II of v-§ 4 tells us that the equations $Z_{ij} = 0$ are equivalent to

$$Z_{\alpha\beta} - \tfrac{1}{2}g_{\alpha\beta}Z_k^k = 0, \quad (49)$$

$$Z_{i|j}^j = 0, \quad (50)$$

with the initial condition

$$Z_i^4 = 0 \text{ on } x^4 = 0. \quad (51)$$

Now (49) is the same as

$$R_{\alpha\beta} = -\kappa(T_{\alpha\beta} - \tfrac{1}{2}g_{\alpha\beta}T_k^k), \quad (52)$$

and (50) is the same as

$$T_{|j}^{ij} = 0. \quad (53)$$

When we substitute (40) in this last and use the Maxwell equations as in (41), we get by (30)

$$(vJ^i)_{|j}J^j - F^{ij}J_j = 0. \quad (54)$$

The initial condition (51) is (by I-§ 9 we know that G_i^4 is CD)

$$G_i^4 + \kappa vJ^4J_i + \kappa E_i^4 = 0 \text{ on } x^4 = 0. \quad (55)$$

We have now before us this initial condition, the equations (52) and (54), and the Maxwell equations

$$F^{ij}_{|j} = J^i, \quad F^{*ij}_{|j} = 0. \quad (56)$$

It is an interesting fact, even apart from its place in the present argument, that (56) is equivalent to

$$F^{\alpha j}{}_{|j} = J^{\alpha}, \quad J^i{}_{|i} = 0, \quad F^{*\alpha j}{}_{|j} = 0, \quad (57)$$

with the initial condition

$$F^{4j}{}_{|j} = J^4, \quad F^{*4j}{}_{|j} = 0 \text{ on } x^4 = 0. \quad (58)$$

(It is easy to see that this condition involves only CD.) Obviously (56) implies (57) and (58). To prove the converse, we have only to show that (58) holds, not merely initially, but permanently. Now in view of the identity

$$F^{ij}{}_{|ij} = 0, \quad (59)$$

(57) implies

$$(F^{ij}{}_{|j} - J^i)_{|i} = 0, \quad (60)$$

and so

$$\begin{aligned} (F^{4j}{}_{|j} - J^4)_{,4} &= - (F^{\alpha j}{}_{|j} - J^{\alpha})_{,\alpha} - \Gamma^i{}_{ki} (F^{kj}{}_{|j} - J^k) \\ &= - \Gamma^i{}_{4i} (F^{4j}{}_{|j} - J^4). \end{aligned} \quad (61)$$

Under the initial condition (58), the only solution is

$$F^{4j}{}_{|j} - J^4 = 0. \quad (62)$$

This establishes the permanence of the first of (58). The permanence of the second is shown in the same way, and thus (56) is equivalent to (57) with (58).

We choose the CD to satisfy (55) and (58) on $x^4 = 0$, and try to solve for the quantities (47) the equations (52), (54) and (57). Now (52) gives $g_{\alpha\beta,44}$ uniquely in terms of CD, and (57) gives

$$F^{\alpha 4}{}_{,4}, \quad J^4{}_{,4}, \quad F^{*\alpha 4}{}_{,4}. \quad (63)$$

Since $F_{\alpha 4}$ and $F^{*}_{\alpha 4}$ comprise all of F_{ij} , it remains only to solve (54) for $\nu_{,4}$ and $J^{\alpha}{}_{,4}$. We obtain

$$\nu_{,4}(J^4)^2 = \text{CD}, \quad \nu J^{\alpha}{}_{,4} J^4 = -\nu_{,4} J^{\alpha} J^4 + \text{CD}. \quad (64)$$

If $J^4 = 0$, the solution for $\nu_{,4}$ is not unique, and so we recognize as a shock wave (characteristic) any 3-space built up of world-lines of current. But if the 3-space $x^4 = 0$ is not such, then $J^4 \neq 0$ and ¹ (64) gives unique solutions for $\nu_{,4}$ and $J^{\alpha}{}_{,4}$, so that the Cauchy problem

¹ We assume $\nu \neq 0$.

is regular, and the chosen CD (subject to the conditions stated) yield a unique solution in the neighbourhood of $x^4 = 0$.

For a vacuum ($J_i = 0$), the above argument is modified and simplified, but we shall not go into it; the Cauchy problem is regular unless $x^4 = 0$ is a null surface.

The case of a vacuum is in some ways more interesting than that of a charged fluid, and we note that the field equations are

$$G_{ij} = -\kappa E_{ij}, \quad F^{ij}{}_{|j} = 0, \quad F^{*ij}{}_{|j} = 0, \quad C_i(g) = 0, \quad (65)$$

with E_{ij} as in (40). Since now

$$E^{ij}{}_{|j} = 0, \quad (66)$$

the count of independent equations in (65) reads $6+3+3+4 = 16$, which agrees with the number of unknowns (g_{ij}, F_{ij}).

Since $E^i_i = 0$, the first of (65) may also be written

$$R_{ij} = -\kappa E_{ij}, \quad (67)$$

and hence

$$R = 0, \quad (68)$$

so that for an electromagnetic field in vacuo the curvature invariant is zero.

§ 3. INTEGRAL ELECTROMAGNETIC THEOREMS

Let X_{ij} be any skew-symmetric tensor and X^*_{ij} its dual, so that, as in (7) and (8),

$$\begin{aligned} X^{*ij} &= \frac{1}{2}\eta^{ijkm}X_{km}, \quad X^*_{ij} = \frac{1}{2}\eta_{ijkm}X^{km}, \\ X^{ij} &= -\frac{1}{2}\eta^{ijkm}X^*_{km}, \quad X_{ij} = -\frac{1}{2}\eta_{ijkm}X^{*km}. \end{aligned} \quad (69)$$

We are to remember that the covariant η_{ijkm} is obtained from the contravariant η^{ijkm} by lowering the superscripts in the usual way by means of g_{ij} . Note that the dual of the dual is the *negative* of the original tensor:

$$X^{**ij} = \frac{1}{2}\eta^{ijkm}X^*_{km} = \frac{1}{4}\eta^{ijkm}\eta_{kmab}X^{ab} = -X^{ij}. \quad (70)$$

The general rule covering changes of sign is that there is a change of sign when star and no-star are interchanged.

From the first of (69),

$$X^{*ij}{}_{|j} = \frac{1}{2}\eta^{ijkm}X_{km|j}. \quad (71)$$

Multiply by η_{iabc} and use (6):

$$\eta_{iabc}X^{*ij}{}_{|j} = -\frac{1}{2}\delta_{abc}^{jkm}X_{km|j} = -(X_{ab|c} + X_{bc|a} + X_{ca|b}). \quad (72)$$

Introducing the symbol $[]$ of cyclic permutation, defined in general by

$$Y_{[abc]} = Y_{abc} + Y_{bca} + Y_{cab}, \quad (73)$$

(72) reads

$$\eta_{iabc}X^{*ij}{}_{|j} = -X_{[ab|c]} = -X_{[ab,c]}. \quad (74)$$

By the rule stated above, we have also the dual formula

$$\eta_{iabc}X^{ij}{}_{|j} = X^*_{[ab|c]} = X^*_{[ab,c]}. \quad (75)$$

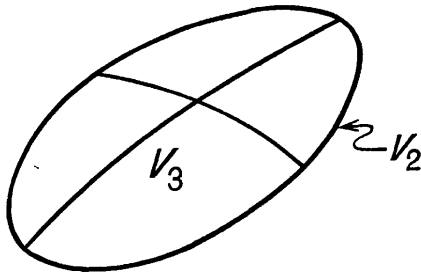


Fig. 1 – Integral electromagnetic theorems

By virtue of these identities it is clear that the first of the Maxwell equations (9) may be written in the equivalent forms

$$F^{ij}{}_{|j} = J^i, \quad F^*_{[ab,c]} = \eta_{iabc}J^i, \quad (76)$$

and the second in the equivalent forms

$$F_{[ij,k]} = 0, \quad F^{*ij}{}_{|j} = 0. \quad (77)$$

The form (10) is thus verified.

Let V_2 be a closed 2-space in space-time, spanned by an open V_3 (Fig. 1). By the theorem of Stokes [I-(241)],

$$\begin{aligned} \oint_{V_2} F_{ij}d\tau^{ij} &= \int_{V_3} F_{ij,k}d\tau^{ijk}, \\ \oint_{V_2} F^*_{ij}d\tau^{ij} &= \int_{V_3} F^*_{ij,k}d\tau^{ijk}. \end{aligned} \quad (78)$$

On account of the skew-symmetry of F_{ij} , F^*_{ij} and $d\tau^{ijk}$, these may be written

$$\begin{aligned} \oint_{V_2} F_{ij}d\tau^{ij} &= \frac{1}{3} \int_{V_3} (F_{ij,k} + F_{jk,i} + F_{ki,j})d\tau^{ijk}, \\ \oint_{V_2} F^*_{ij}d\tau^{ij} &= \frac{1}{3} \int_{V_3} (F^*_{ij,k} + F^*_{jk,i} + F^*_{ki,j})d\tau^{ijk}. \end{aligned} \quad (79)$$

In these integrals F_{ij} is arbitrary. If we now impose Maxwell's

equations, we get two integral electromagnetic theorems as follows:

$$\oint_{V_2} F_{ij} d\tau^{ij} = 0, \quad (80)$$

$$\oint_{V_2} F_{ij}^* d\tau^{ij} = \frac{1}{3} \int_{V_3} \eta_{aijk} J^a d\tau^{ijk}. \quad (81)$$

This is probably the neatest form in which to express these results. But, to bring them closer to familiar ideas, we may write, as in I-(247) and I-(249),

$$\begin{aligned} d\tau^{ij} &= \varepsilon(M)\varepsilon(N)\eta^{ijkm}M_kN_md_2v, \\ d\tau^{ijk} &= \varepsilon(L)\eta^{ijkm}L_md_3v; \end{aligned} \quad (82)$$

here M^i and N^i are unit vectors, orthogonal to V_2 and to one another, and L^i is a unit vector orthogonal to V_3 ; d_2v and d_3v are invariant elements of area and 3-volume respectively. Then

$$\begin{aligned} F_{ij}d\tau^{ij} &= 2\varepsilon(M)\varepsilon(N)F^{*km}M_kN_md_2v, \\ F_{ij}^*d\tau^{ij} &= -2\varepsilon(M)\varepsilon(N)F^{km}M_kN_md_2v. \end{aligned} \quad (83)$$

Noting that by (6)

$$\eta_{aijk}\eta^{ijk m} = 6\delta_a^m, \quad (84)$$

we have

$$\frac{1}{3}\eta_{aijk}J^a d\tau^{ijk} = 2\varepsilon(L)L_iJ^i d_3v. \quad (85)$$

Thus the formulae (80) and (81) may be written

$$\oint_{V_2} \varepsilon(M)\varepsilon(N)F_{km}^*M^kN^m d_2v = 0, \quad (86)$$

$$\oint_{V_2} \varepsilon(M)\varepsilon(N)F_{km}M^kN^m d_2v = -\int_{V_3} \varepsilon(L)L_iJ^i d_3v. \quad (87)$$

Note the interchange of F and F^* in passing from the earlier forms. It is of course understood that proper attention is paid to the orientation of the vectors L^i , M^i , N^i [cf. I-§ 10].

Just as we defined invariant components in (2) we can define starred invariant components,

$$F_{(ab)}^* = F_{ij}^*\lambda_{(a)}^i\lambda_{(b)}^j = -F_{(ba)}^*. \quad (88)$$

To evaluate these in terms of the invariants E_α , H_α of (3), we note

that (69) gives ($q = \sqrt{-g}$)

$$F^{*23} = q^{-1}F_{14}, \quad F^{*14} = q^{-1}F_{23}, \quad F_{23}^* = -qF^{14}, \quad F_{14}^* = -qF^{23}, \quad (89)$$

and eight other equations obtained from these by cyclic permutation of 1, 2, 3. If we take special coordinates making locally

$$g_{\alpha\beta} = \delta_{\alpha\beta}, \quad g_{44} = -1, \quad \lambda_{(a)}^i = \delta_a^i,$$

then

$$\begin{aligned} F_{(23)}^* &= F_{23}^* = -F^{14} = F_{14} = F_{(14)} = E_1, \\ F_{(14)}^* &= F_{14}^* = -F^{23} = -F_{23} = -F_{(23)} = -H_1. \end{aligned} \quad (90)$$

Thus, in general coordinates, the invariant components are

$$\begin{aligned} F_{(14)}^* &= -H_1, & F_{(24)}^* &= -H_2, & F_{(34)}^* &= -H_3, \\ F_{(23)}^* &= E_1, & F_{(31)}^* &= E_2, & F_{(12)}^* &= E_3. \end{aligned} \quad (91)$$

The formulae (80), (81), (86), (87) are very general — there is no restriction to timelike or spacelike V_3 . As a simple illustration let V_3 be spacelike, so that L^i is timelike, and let us choose over V_3 a system of orthonormal tetrads with $\lambda_{(4)}^i = L^i$, making $\lambda_{(4)}^i = M^i$, $\lambda_{(1)}^i = N^i$ on the boundary V_2 , so that $\lambda_{(1)}^i$ is the unit normal to V_2 in V_3 . Then (86) and (87) give

$$\oint_{V_2} H_1 d_2 v = 0, \quad (92)$$

$$\oint_{V_2} E_1 d_2 v = \int_{V_3} J_{(4)} d_3 v. \quad (93)$$

Here E_1 and H_1 are normal components of the electric and magnetic vectors. We recognise the *theorem of Gauss*: (92) says that the normal flux of the magnetic vector vanishes for a closed surface, and (93) equates the normal flux of the electric vector to the total contained charge (a factor 4π does not appear on account of the rational units used).

To illustrate the concept of ‘closed 2-space’ in familiar terms, the simplest example one can think of is the instantaneous existence of a spherical surface. A harder one is the 2-space generated by the history of a closed loop of wire, at rest during a finite time. A closed V_2 is formed by the history of the wire (this part is timelike) and the instantaneous existence of a fictitious membrane stretched across the wire at the first instant and the last (this part is spacelike).

In addition to the above integral electromagnetic theorems, there is a much simpler one, viz.

$$\oint_{V_3} \varepsilon(N) J_i N^i d_3v = 0, \quad (94)$$

with the integral taken over any closed 3-space, N^i being its unit normal. This may be said to express the conservation of electric charge. It is an immediate consequence of $J^i{}_{|i} = 0$.

§ 4. ELECTROVAC UNIVERSES

Consider a universe consisting of an interior domain I (which may consist of several world-tubes) and an exterior domain E . There is no matter in E , but there is an electromagnetic field, and to emphasize this we may speak of an *electrovac* universe. In E we have a symmetric metric tensor g_{ij} and a skew-symmetric electromagnetic tensor F_{ij} , nothing more, satisfying the field equations

$$G_{ij} = -\kappa E_{ij}, \quad \kappa = 8\pi, \quad (95)$$

and Maxwell's equations

$$F^i{}_{|j} = 0, \quad F^{*ij}{}_{|j} = 0, \quad (96)$$

where

$$E_{ij} = g^{ab} F_{ai} F_{bj} - \frac{1}{4} g_{ij} F_{ab} F^{ab}. \quad (97)$$

Since $E^i{}_i = 0$, the field equations (95) are equivalent to

$$R_{ij} = -\kappa E_{ij}. \quad (98)$$

With regard to I , we keep an open mind.

Let us now specialize to a statical universe by taking the metric form to be, as in VIII-(161),

$$\Phi = g_{\alpha\beta} dx^\alpha dx^\beta - V^2 (dx^4)^2, \quad (99)$$

with coefficients independent of x^4 . We satisfy the second of (96) by writing

$$F_{ij} = \phi_{j,i} - \phi_{i,j}. \quad (100)$$

Let us choose $\phi_\alpha = 0$ and take ϕ_4 to be independent of x^4 . Then, putting $\phi_4 = \phi$ for simplicity of writing, we have

$$F_{\alpha 4} = \phi_{,\alpha}, \quad F_{\alpha\beta} = 0. \quad (101)$$

The physical interpretation of this situation is realistic. We may think of a single massive body carrying an electric charge, or (somewhat less realistically) of several such bodies, their gravitational

attraction being balanced by electrostatic repulsions. By interchanging the roles of F_{ij} and F_{ij}^* , we get the case of a magnetized massive body.

To explore E , we may use the calculations of VIII-§ 5, a bar indicating quantities pertaining to space. By (101) we have

$$F^{\alpha 4} = -V^{-2} \bar{g}^{\alpha\beta} \phi_{,\beta}, \quad F^{\alpha\beta} = 0, \quad (102)$$

and by (97)

$$\begin{aligned} E_{\alpha\beta} &= V^{-2} (\tfrac{1}{2} \bar{g}_{\alpha\beta} \Delta_1 \phi - \phi_{,\alpha} \phi_{,\beta}), \\ E_{\alpha 4} &= 0, \quad E_{44} = \tfrac{1}{2} \Delta_1 \phi. \end{aligned} \quad (103)$$

Then, by VIII-(164), the field equations read

$$R_{\alpha\beta} \equiv \bar{R}_{\alpha\beta} + V^{-1} V_{\parallel\alpha\beta} = \kappa V^{-2} (\phi_{,\alpha} \phi_{,\beta} - \tfrac{1}{2} \bar{g}_{\alpha\beta} \Delta_1 \phi), \quad (104)$$

$$-R_{44} \equiv V \Delta_2 V = \tfrac{1}{2} \kappa \Delta_1 \phi, \quad (105)$$

the double vertical stroke indicating covariant differentiation with respect to $\bar{g}_{\alpha\beta}$ ($= g_{\alpha\beta}$). We have by (11)

$$F^i{}_{|j} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^j} (\sqrt{-g} F^{ij}) = \frac{1}{V \sqrt{\bar{g}}} \frac{\partial}{\partial x^\beta} (V \sqrt{\bar{g}} F^{i\beta}), \quad (106)$$

and so all the Maxwell equations (96) are satisfied identically except one which reads

$$V \Delta_2 \phi - \bar{g}^{\alpha\beta} V_{,\alpha} \phi_{,\beta} = 0. \quad (107)$$

Our general problem is then to find the eight quantities $\bar{g}_{\alpha\beta}$, V , ϕ to satisfy the eight equations in (104), (105) and (107).

At this point we confine our attention to those solutions in which V is a function of ϕ , i.e. the level surfaces of V and ϕ coincide¹. Then writing $dV/d\phi = V'$, $d^2V/d\phi^2 = V''$, we have

$$\begin{aligned} V_{,\alpha} &= V' \phi_{,\alpha}, & V_{\parallel\alpha\beta} &= V' \phi_{\parallel\alpha\beta} + V'' \phi_{,\alpha} \phi_{,\beta} \\ \Delta_1 V &= V'^2 \Delta_1 \phi, & \Delta_2 V &= V' \Delta_2 \phi + V'' \Delta_1 \phi, \end{aligned} \quad (108)$$

and (105) and (107) become

$$VV' \Delta_2 \phi + (VV'' - \tfrac{1}{2} \kappa) \Delta_1 \phi = 0, \quad (109)$$

$$V \Delta_2 \phi - V' \Delta_1 \phi = 0. \quad (110)$$

¹ Cf. WEYL [1917], MAJUMDAR [1946], [1947], PAPAPETROU [1947], BONNOR [1953], [1954a]. I am much indebted to Mr. A. Das for information and discussions of this work.

Eliminating the first terms from these two equations, and noting that $\Delta_1\phi \neq 0$ (otherwise the electromagnetic field would vanish), we obtain for $V(\phi)$ the differential equation

$$VV'' + V'^2 = \frac{1}{2}\kappa. \quad (111)$$

The general solution is

$$V^2 = A + B\phi + \frac{1}{2}\kappa\phi^2, \quad (112)$$

where A and B are arbitrary constants. With this choice of $V(\phi)$, we have in (104) and (110) seven equations for the seven quantities $\bar{g}_{\alpha\beta}$, ϕ .

The theory becomes more interesting if we specialize the general statcal form (99) to the particular conformastat form

$$\Phi = U^2 dx^\alpha dx^\alpha - U^{-2} (dx^4)^2, \quad (113)$$

so that

$$\bar{g}_{\alpha\beta} = U^2 \delta_{\alpha\beta}, \quad V = U^{-1}, \quad \bar{g} = U^6. \quad (114)$$

Instead of assuming at once that U is a function of ϕ , it is more illuminating to apply directly to (113) the field equations (104), (105) and the Maxwell equation (107). But before doing that, we recall that the field equations (98) imply $R = 0$. Referring to VIII-(181), we see that this implies

$$U_{\sigma\sigma} = 0. \quad (115)$$

(We now denote partial derivatives by subscripts without commas.) This formula is the key to the situation — U is *harmonic with respect to the flat metric* $dx^\alpha dx^\alpha$. (Contrast this electrovac result with the vacuum case for which, not U , but \sqrt{U} is harmonic, as in VIII-(177).)

Remembering that the operators Δ_1 and Δ_2 are taken with respect to $\bar{g}_{\alpha\beta}$, we have

$$\Delta_1\phi = U^{-2}\phi_\sigma\phi_\sigma, \quad \bar{g}_{\alpha\beta}\Delta_1\phi = \delta_{\alpha\beta}\phi_\sigma\phi_\sigma, \quad (116)$$

and, since $\sqrt{\bar{g}} = U^3$,

$$\begin{aligned} \Delta_2\phi &= U^{-3}(U^3\bar{g}^{\alpha\beta}\phi_\beta)_\alpha = U^{-3}(U\phi_\alpha)_\alpha = U^{-2}\phi_{\alpha\alpha} + U^{-3}U_\alpha\phi_\alpha, \\ \bar{g}^{\alpha\beta}V_\alpha\phi_\beta &= -U^{-4}U_\alpha\phi_\alpha, \quad V\Delta_2V = U^{-6}U_\sigma U_\sigma - U^{-5}U_{\sigma\sigma}. \end{aligned} \quad (117)$$

Thus, with the aid of VIII-(181), the field equations (104), (105) read

$$U_\alpha U_\beta - \frac{1}{2}\delta_{\alpha\beta}U_\sigma U_\sigma = \frac{1}{2}\kappa U^4(\phi_\alpha\phi_\beta - \frac{1}{2}\delta_{\alpha\beta}\phi_\sigma\phi_\sigma), \quad (118)$$

$$U_\sigma U_\sigma - UU_{\sigma\sigma} = \frac{1}{2}\kappa U^4\phi_\sigma\phi_\sigma, \quad (119)$$

and Maxwell's equation (107) reads

$$U\phi_{\sigma\sigma} + 2U_{\sigma}\phi_{\sigma} = 0. \quad (120)$$

Now the remarkable fact emerges that all *eight* equations contained in (118), (119) and (120) are satisfied if we put ¹

$$\phi = \sqrt{\frac{2}{\kappa}} \frac{1}{U} = \frac{1}{\sqrt{4\pi}} \frac{1}{U}, \quad U_{\sigma\sigma} = 0. \quad (121)$$

Thus we have a very simple way of constructing an electrovac field: *Choose any harmonic function U which has no zero in E and define ϕ by (121).* To get the standard flat metric at infinity, we should choose U so that U^2 tends to unity at infinity.

Suppose now that the interior domain I consists of a number of separate parts, I_1, I_2, \dots . Round any one of these, say I_1 , draw a closed surface S_1 . Since $J^i = 0$ in E , we know from (93) that there is a certain integral taken over S_1 which does not change when we deform S_1 in E . Without troubling about the internal structure of I_1 , we naturally define the *total charge* e_1 in or on it by

$$e_1 = \int_{S_1} E_1 d_2v, \quad (122)$$

in the notation of (93). In dealing with this integral, we must be careful to distinguish between physical metric

$$d\sigma^2 = U^2 dx^\alpha dx^\alpha \quad (123)$$

and the flat metric

$$d\sigma_0^2 = dx^\alpha dx^\alpha. \quad (124)$$

The integral (122) is set up in the former. By (3)

$$E_1 = F_{(14)} = F_{ij}\lambda_{(1)}^i\lambda_{(4)}^j \quad (125)$$

where $\lambda_{(1)}^i$ is the outward unit normal to S_1 and $\lambda_{(4)}^i$ a unit vector in the time-direction, and by (101) this reduces to

$$E_1 = \phi_\alpha \lambda_{(1)}^\alpha \lambda_{(4)}^4. \quad (126)$$

The unit vectors here are of course unit with respect to $d\sigma$. Thus if n^α is the outward normal to S_1 which is unit with respect to $d\sigma_0$, we have

$$\lambda_{(1)}^\alpha = \frac{dx^\alpha}{d\sigma} = \frac{dx^\alpha}{d\sigma_0} \frac{d\sigma_0}{d\sigma} = n^\alpha U^{-1}, \quad \lambda_{(4)}^4 = U. \quad (127)$$

If dS is an element of area with respect to $d\sigma_0$, we have $d_2v = U^2 dS$,

¹ Note that, since $UV = 1$, this is the same as (112) with $A = B = 0$.

and (122) gives for the charge associated with I_1 the value

$$e_1 = \int_{S_1} U^2 \phi_\alpha n^\alpha dS = - \frac{1}{\sqrt{4\pi}} \int_{S_1} U_\alpha n^\alpha dS. \quad (128)$$

Since U is harmonic with respect to $d\sigma_0$, this last integral is obviously unchanged by deformation of S_1 ; that we could have stated at once, but we needed to connect the integral with the charge.

To complete the electrovac model we need to fill in a metric in I , but that will not be discussed here.

Let us take a set of points P_1, P_2, \dots and define I to consist of the interiors of spheres of radii a_1, a_2, \dots centred at these points. Let ρ_1, ρ_2, \dots be the distances (measured by $d\sigma_0$) of a general point P from the centres. Write

$$U(P) = \varepsilon + \frac{1}{\sqrt{4\pi}} \left(\frac{e_1}{\rho_1} + \frac{e_2}{\rho_2} + \dots \right), \quad \varepsilon = \pm 1. \quad (129)$$

Provided the ratios $e_1/a_1, e_2/a_2, \dots$ are small enough, U is a harmonic function which does not vanish in E ; the potential is

$$\phi = \frac{1}{\varepsilon \sqrt{4\pi} + \frac{e_1}{\rho_1} + \frac{e_2}{\rho_2} + \dots} \quad (130)$$

and the charges associated with the spheres are e_1, e_2, \dots . Provided the metric can be filled in suitably in I , we have before us a set of charged bodies, in equilibrium under their mutual interactions. It is interesting to note that, on account of the ambiguity in ε , there seem to be two fields corresponding to given charges. If we reverse the signs of ε and of all charges, we leave the metric unchanged but reverse the electric field. As remarked earlier, there is no difficulty in changing from electricity to magnetism, and we might replace (129) by a formula corresponding to a set of magnetic dipoles. The theory rests on the fact that, for the special conformastat metric (113), $R = 0$ implies that U is harmonic with respect to the flat metric $d\sigma_0$.

CHAPTER XI

GEOMETRICAL OPTICS

§ 1. WAVE-KINEMATICS IN SPACE-TIME

Consider a single infinity of 3-spaces in space-time (Fig. 1); they may be spacelike, null or timelike, by which we mean that their normals are timelike, null or spacelike, respectively. We call these 3-spaces *3-waves*, or briefly *waves*. We associate with each wave a *phase-angle* ϕ , increasing monotonically as we run from wave to wave, so that we may

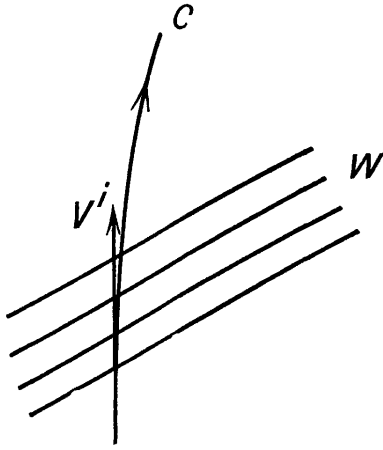


Fig 1 – Phase waves W and observer C

refer to the waves as *phase-waves*. Those waves for which $\phi = 2n\pi$ will be called *crests*; we may regard the waves drawn in Fig. 1 as crests.

Since ϕ is a function of position in space-time, we may write

$$f(x) = -\frac{h}{2\pi} \phi, \quad (1)$$

so that the equations of the waves are $f(x) = \text{const.}$; h is a small universal constant, regarded as infinitesimal for mathematical purposes. Any small constant would do, but for certain reasons it is desirable to take h to be Planck's constant. In passing from crest to crest, $f(x)$ changes by

$$df = -h. \quad (2)$$

Let C be the timelike world-line of an observer with 4-velocity V^i . We seek expressions for (i) the frequency of the waves, and (ii) their speed, both relative to C .

Let us write

$$p_i = f_{,i}. \quad (3)$$

In passing from a crest to the next crest along C , with a displacement

dx^i and time-element ds on C , we have by (2)

$$p_i dx^i = p_i V^i ds = -h, \quad (4)$$

and so the period τ ($= ds$) and the frequency ν ($= 1/\tau$) are

$$\tau = -\frac{h}{p_i V^i}, \quad \nu = -\frac{p_i V^i}{h}. \quad (5)$$

These are invariants, but their values depend on C . We note that

$$h\nu = -p_i V^i, \quad (6)$$

and, if we introduce an orthonormal tetrad $\lambda_{(a)}^i$ with $\lambda_{(4)}^i$ along V^i , then

$$h\nu = -p_{(4)} = p^{(4)}. \quad (7)$$

We call p^i the *frequency 4-vector* of the waves. The minus sign was used in (1) in order to get $+p^{(4)}$ in (7).

In (5) the expression for ν is ugly — a finite quantity divided by an infinitesimal. But we are in fact concerned with waves of high frequency, this being the usual condition under which geometrical optics is physically valid. It may be remarked, however, that there is nothing essentially *optical* about the kinematics of the present section — it applies equally to elastic or other waves of high frequency.

We have dealt with frequency first because it is so simple. But the question of speed is really more general, because only a single wave is involved, and we do not have to think about phase. Let A (Fig. 2) be the intersection of C with a wave W . We think of a fictitious particle which rides on the wave, so that its world-line lies in W , and for W we take the equation $f(x) = \text{const.}$ as above. Let AB (dx^i) be an infinitesimal displacement on this particle's world-line; let NB ($d\xi^i$) be orthogonal to C ; and let $AN = ds$. Then the speed u' of the fictitious particle relative to C is naturally defined by

$$u' = \frac{NB}{AN}, \quad u'^2 = \frac{d\xi_i d\xi^i}{ds^2}. \quad (8)$$

(We write $dx_i = g_{ij}dx^j$, $d\xi_i = g_{ij}d\xi^j$.)

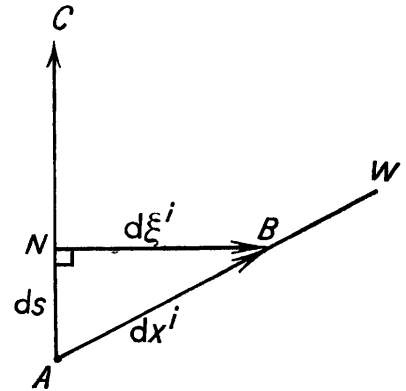


Fig. 2 — Speed of wave W relative to observer C

From the construction in Fig. 2 we have (with $p_i = f_{,i}$ as earlier)

$$p_i dx^i = 0, \quad (9)$$

$$V^i d\xi^i = 0, \quad (10)$$

$$d\xi^i = dx^i - V^i ds. \quad (11)$$

From (10) and (11), since $V_i V^i = -1$,

$$ds = -V_i dx^i, \quad (12)$$

and so (11) becomes

$$d\xi^i = dx^i + V^i V_j dx^j. \quad (13)$$

Thus

$$d\xi_i d\xi^i = dx_i dx^i + (V_i dx^i)^2, \quad (14)$$

which is the theorem of Pythagoras with attention paid to the indefinite character of the metric. From (8), (12) and (14), the speed of the fictitious particle is given by

$$u'^2 = 1 + \frac{dx_i dx^i}{(V_j dx^j)^2}. \quad (15)$$

An examination of what we mean in ordinary physics by the speed of a wave suggests that we should define the speed (u) of the wave as the *minimum* value of u' for all fictitious particles riding on it. Thus we are to minimize (15) with (9) as a side condition, and this gives

$$dx^i = \alpha V^i + p^i, \quad (16)$$

where α is a Lagrange multiplier (we omit a multiplier in front of p^i since only the ratios of dx^i are involved in (15)). By (9) we get

$$\alpha = -\frac{p_i p^i}{p_j V^j}, \quad (17)$$

and (16) gives

$$dx_i dx^i = \alpha(p_i V^i - \alpha), \quad V_i dx^i = p_i V^i - \alpha. \quad (18)$$

From (15) we then obtain for the square of the *speed of the wave* relative to C the formula

$$u^2 = \frac{p_j V^j}{p_i V^i - \alpha}. \quad (19)$$

In dealing with waves, the *slowness* (u^{-1}) is more fundamental than the speed (u). For the square of the slowness we have

$$\frac{1}{u^2} = 1 + \frac{p_i p^i}{(p_j V^j)^2}. \quad (20)$$

This invariant formula contains only the 4-velocity of the observer and the frequency 4-vector. But, as remarked above, the idea of *frequency* is not really involved — we may replace p^i by any vector normal to the wave.

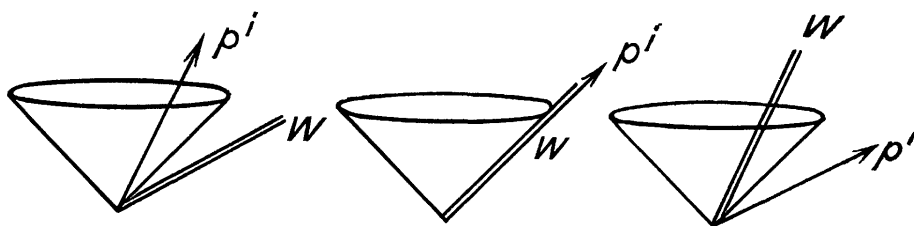


Fig. 3 – Waves W and frequency vectors p^i , with the null cone

By (20) the slowness is less or greater than unity according as p^i is timelike or spacelike. Accordingly we have the following classification of waves, illustrated in Fig. 3:

wave	frequency vector	speed
spacelike	timelike ($p_i p^i < 0$)	$u > 1$
null	null ($p_i p^i = 0$)	$u = 1$
timelike	spacelike ($p_i p^i > 0$)	$u < 1$

In ordinary parlance, $u > 1$ means ‘faster than light’, and $u < 1$ ‘slower than light’, but these expressions must be used here with great caution, because the waves in question may themselves be light waves. The true comparison is with the null cone, which represents a fundamental type of shock wave.

§ 2. WAVES, RAYS AND PHOTONS IN A DISPERSIVE MEDIUM

Although the following theory has wider physical applications, we shall use the language of optics. It is essentially a physical transcript, with interpretations, of the Hamiltonian theory discussed in I-§ 7 and the theory of characteristics of v-§ 7, which is really part of Hamiltonian theory.

Consider a transparent medium specified by its 4-velocity V^i and certain physical properties, such as density, which we shall include in a symbol ρ . We regard space-time as given, so that g_{ij} , V^i and ρ are given functions of the coordinates. Our object is to set up a theory of geometrical optics in the medium on the basis of reasonable hypotheses suggested by classical theory.

In classical optics, we describe a medium ¹ by giving the refractive index n as a function of frequency and other local properties, included in the symbol ρ above. Now the refractive index is the reciprocal of the phase-speed, and so we are led by (20) to base relativistic geometrical optics on a *medium-equation*

$$n^2 = 1 + \frac{p_i p^i}{(p_j V^j)^2}, \quad (21)$$

where n is the refractive index ($n = u^{-1}$), a *given* function of the coordinates x^i and of $p_k V^k$ ($= h\nu$, where ν is the frequency). Note that, whereas in § 1 V^i was the 4-velocity of an arbitrary observer, it is now that of the medium, so that phase-speed and frequency are measured in the instantaneous rest frame of the medium.

To apply Hamiltonian methods, it is necessary to use p_i rather than p^i in writing the medium-equation, and we shall express this equation in the form

$$\omega(x, p) = 0, \quad (22)$$

where

$$\omega(x, p) = \frac{1}{2}[g^{ij}p_i p_j - (n^2 - 1)(p_i V^i)^2]. \quad (23)$$

This may also be written

$$\omega(x, p) = \frac{1}{2}\bar{g}^{ij}p_i p_j, \quad (24)$$

where

$$\bar{g}^{ij} = g^{ij} - (n^2 - 1)V^i V^j. \quad (25)$$

It is easy to see that the conjugate covariant tensor \bar{g}_{ij} , defined by

$$\bar{g}_{ij}\bar{g}^{ik} = \delta_j^k, \quad (26)$$

is

$$\bar{g}_{ij} = g_{ij} + \left(1 - \frac{1}{n^2}\right)V_i V_j. \quad (27)$$

¹ We consider only isotropic media. A relativistic treatment of anisotropic media would be very complicated.

It might appear that this provides space-time with a new metric tensor, but that is not the case since n depends on p_i . But for a *non-dispersive* medium n is a function of position only, and then indeed \bar{g}_{ij} can be regarded as a second metric tensor; this idea will be used later.

Since $p_i = f_{,i}$ as in (3), we have in (22) a partial differential equation of the first order for the phase-function $f(x)$. As in v-(153), the characteristic curves of this equation satisfy

$$\frac{dx^i}{dw} = \frac{\partial \omega}{\partial p_i}, \quad \frac{dp_i}{dw} = -\frac{\partial \omega}{\partial x^i}, \quad (28)$$

where w is a parameter; these characteristic curves are *optical rays*, and the totality of all possible rays consists of the solutions of these ordinary differential equations, a solution being determined by an arbitrary initial event (x) and an initial frequency vector (p) which is arbitrary save for (22). Note that (28) gives, not only a ray, but also a frequency vector at each point on it.

By (23)

$$\frac{\partial \omega}{\partial p_i} = p^i - (n^2 - 1)(p_j V^j) V^i - nn'(p_j V^j)^2 V^i, \quad (29)$$

where n' is the partial derivative of n with respect to $p_k V^k$. Thus by (28) the direction of the ray lies in the 2-element defined by the frequency vector and the 4-velocity of the medium, as indeed we would expect for isotropy. In general the ray does *not* point along the frequency vector, but it does so in vacuo, for then $n = 1$.

If we start with any small piece of a phase-wave, the construction given in v-§ 7 carries this piece along the rays, with changing phase in general, there being an 'action element' $p_i dx^i = -h$ between successive crests. We recognize this advancing element as a *signal* (transmission of information or energy), and so it appears essential that *the rays should be timelike (or null)*, since otherwise we would violate concepts of causality. Mathematically, this requirement reads

$$g_{ij} \frac{\partial \omega}{\partial p_i} \frac{\partial \omega}{\partial p_j} \leq 0. \quad (30)$$

Whether a ray be timelike, null or spacelike, we may define the *ray-speed* v (relative to the medium) by following an event along the ray as in Fig. 2, understanding W to indicate the ray. A simple

calculation based on (10), (11) and (28) gives

$$v^2 = 1 + g_{ij} \frac{\partial \omega}{\partial p_i} \frac{\partial \omega}{\partial p_j} \left(V_k \frac{\partial \omega}{\partial p_k} \right)^{-2}, \quad (31)$$

so that the inequality (30) is equivalent to $v \leq 1$. On substitution for ω from (23), direct calculation gives

$$g_{ij} \frac{\partial \omega}{\partial p_i} \frac{\partial \omega}{\partial p_j} = n^2 (p_i V^i)^2 (1 - q^{-2}), \quad V_i \frac{\partial \omega}{\partial p_i} = n (p_i V^i) q^{-1}, \quad (32)$$

where q is defined by

$$q = n + n' p_i V^i = n + v \frac{\partial n}{\partial v} = \frac{\partial}{\partial v} (nv). \quad (33)$$

When we substitute from (32) in (31), we obtain simply $v = q$. The point of all this is that (33) is the ordinary definition of *group-speed*¹, and we have established its identity with the ray-speed, defined in terms of the characteristic curves of the Hamilton-Jacobi equation obtained by putting $f_{,i}$ for p_i in $\omega(x, p) = 0$. Thus (30) merely reasserts a demand commonly made in physics — *the group-speed cannot exceed the fundamental speed* (that of light in vacuo).

If equality holds in (30), the ray is null. If inequality, the ray is timelike and its unit 4-vector defines the *ray 4-velocity*¹.

A system of rays associated with phase-waves form a coherent system in the sense of I-§ 7, so that

$$\oint p_i dx^i = 0 \quad (34)$$

for every reducible closed circuit in space-time. For such a system, we have before us a mental space-time picture of phase-waves, frequency vectors (normal to the waves), and rays (timelike or null for causal reasons, but not in general normal to the waves).

Where does the *photon*² fit into this picture? In view of (6), it seems appropriate to take the frequency vector p^i to be the 4-momentum of a photon associated with a system of waves, and the history of a photon to be a ray. For the photon in vacuo, we have earlier made some natural assumptions: (i) its world-line is a null geodesic, and

¹ Cf. SYNGE [1954a], [1956b], and § 4 of the present chapter. For a discussion of phase-speed, group-speed and signal-speed, see L. BRILLOUIN [1960].

² In order to attach physical meaning to null geodesics, even inside matter, in III-§ 3 the word 'photon' was restricted to mean one of very high energy. That restriction is of course now withdrawn.

A form a cone¹ (represented in Fig. 4 by a single ray). As we move A along Γ , we get a single infinity of such cones, and thus we fill space-time with rays and a field $p_i(x)$. Using the method of I-§ 7, it is now easy to get the phase-waves. If there is a crest at A ($\phi = 0$), we get the complete crest (wave of zero phase) by finding all events B which satisfy

$$\int_A^B p_i dx^i = 0. \quad (38)$$

On account of (34), it does not matter what path is used (paths of integration are shown by broken lines in Fig. 4, and phase-waves by heavy lines). The n th crest after this one has the equation

$$\int_A^B p_i dx^i = -nh. \quad (39)$$

§ 3. VARIATIONAL PRINCIPLES IN GEOMETRICAL OPTICS

In classical geometrical optics and in classical dynamics, one sets down certain basic equations and develops other equations from these. When a whole coherent body of theory has been created, one realizes that the point of departure for its logical development is largely a matter of taste, for the same structure might have been set on many different bases. But to avoid mental confusion, one must select *some* base and not change it in the course of the argument.

Hamilton based his geometrical optics on Fermat's principle and his dynamics on Newton's equations of motions, and these were the best bases from the standpoint of physical plausibility. In modern times there has been a strong tendency to give pride of place to variational principles. This suggests that we should base relativistic geometrical optics on Fermat's principle. However in a theory which is to include dispersion, a suitable simple form of Fermat's principle is not available, and it has seemed best to start, as in the preceding sections, from the concept of *waves* rather than *rays*. We saw that the phase-function satisfies a certain partial differential equation, written $\omega(x, p) = 0$, and we defined the rays as its characteristics. We shall continue to use that basis for the theory, but develop equivalent variational principles which might, if one so desired, be used as bases for geometrical optics.

¹ In the case of Čerenkov radiation, Γ lies outside the cone formed by the rays.

Consider the following variational principle:

$$\text{Principle } A: \quad \delta \int p_i dx^i = 0, \quad \omega(x, p) = 0. \quad (40)$$

Here the curves considered join a pair of fixed events and p_i is arbitrary along each of them except for the side-condition shown. As in 1-§ 7, Principle A is equivalent to the differential equations

$$\frac{dx^i}{dw} = \frac{\partial \omega}{\partial p_i}, \quad \frac{dp_i}{dw} = -\frac{\partial \omega}{\partial x^i}, \quad (41)$$

w being a special parameter. Since these are precisely the ray-equations (28), we recognize that optical rays satisfy A , and might have been defined by A . Had we adopted this course, the associated waves would have been defined by the method of 1-§ 7.

Before proceeding to a second variational principle, let us carry out some formal work, in which $\omega(x, p)$ is any function for which

$$\det \frac{\partial^2 \omega}{\partial p_i \partial p_j} \neq 0. \quad (42)$$

It must be clearly understood that here we are dealing with a *function* ω and not with an *equation* $\omega = 0$.

Define

$$z^i = \frac{\partial \omega}{\partial p_i} \quad (43)$$

and

$$L = p_j \frac{\partial \omega}{\partial p_j} - \omega. \quad (44)$$

In view of (42), we can solve (43) for the p 's, obtaining $p_i = p_i(x, z)$, and, when we substitute these in (44), we get $L = L(x, z)$. We seek the partial derivatives ¹ of L .

The quantities (x, p) may be varied arbitrarily, the variations in the z 's then following from (43). Thus

$$\frac{\partial L}{\partial x^i} \delta x^i + \frac{\partial L}{\partial z^i} \delta z^i = p_j \frac{\partial^2 \omega}{\partial p_j \partial x^i} \delta x^i + p_j \frac{\partial^2 \omega}{\partial p_j \partial p_i} \delta p_i - \frac{\partial \omega}{\partial x^i} \delta x^i \quad (45)$$

¹ We are going through the same type of argument as that by which, in classical dynamics, one passes from a Hamiltonian to a Lagrangian. It is desirable to give it in detail, because in the classical argument there is an absolute parameter t , not present here.

becomes an identity in the differentials $(\delta x, \delta p)$ when we substitute

$$\delta z^i = \frac{\partial^2 \omega}{\partial p_i \partial x^k} \delta x^k + \frac{\partial^2 \omega}{\partial p_i \partial p_k} \delta p_k. \quad (46)$$

Hence

$$\frac{\partial L}{\partial x^i} + \frac{\partial L}{\partial z^j} \frac{\partial^2 \omega}{\partial p_j \partial x^i} = p_j \frac{\partial^2 \omega}{\partial p_j \partial x^i} - \frac{\partial \omega}{\partial x^i}, \quad (47)$$

$$\left(\frac{\partial L}{\partial z^j} - p_j \right) \frac{\partial^2 \omega}{\partial p_j \partial p_i} = 0. \quad (48)$$

From (48) and (42) we get

$$\frac{\partial L}{\partial z^i} = p_i, \quad (49)$$

and (47) then gives

$$\frac{\partial L}{\partial x^i} = - \frac{\partial \omega}{\partial x^i}. \quad (50)$$

The above is a formal procedure by which, starting from any function $\omega(x, p)$ satisfying (42), we generate a function $L(x, z)$ and its partial derivatives.

We now pass to optics. In Fig. 5, C is a ray joining events P, Q , so that the equations (41) are satisfied, the parameter w running from w_1 to w_2 , say. The ray C belongs to a family of curves joining P and Q , represented by D in Fig. 5.

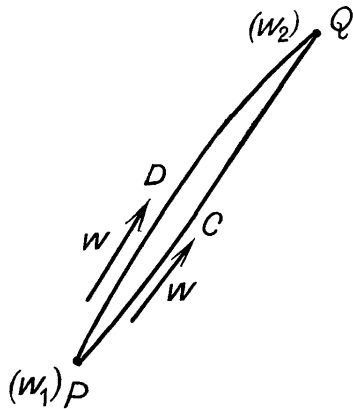


Fig. 5 – The variational principle B

The first step is to put a parameter w on D , arbitrarily except that it is to have the end-values w_1, w_2 . Writing $x^{i'} = dx^i/dw$, we now have a vector field $x^{i'}$ defined on C and on D . On C we have, by (41),

$$x^{i'} = \frac{\partial \omega}{\partial p_i}. \quad (51)$$

The trick is to use this equation to define a vector field p_i on D also. But this is precisely the equation (43), with $x^{i'}$ instead of z^i , and so we generate on C and on D a function $L(x, x')$ with the partial derivatives

$$\frac{\partial L}{\partial x^{i'}} = p_i, \quad \frac{\partial L}{\partial x^i} = - \frac{\partial \omega}{\partial x^i}. \quad (52)$$

The integral $\int L(x, x')dw$ is then meaningful on C and on D . On C we have

$$\frac{d}{dw} \frac{\partial L}{\partial x^{i'}} - \frac{\partial L}{\partial x^i} = \frac{dp_i}{dw} + \frac{\partial \omega}{\partial x^i} = 0, \quad (53)$$

by (41). But these are the well known Euler-Lagrange equations for the variational principle

$$\text{Principle } B: \quad \delta \int L(x, x')dw = 0, \quad (54)$$

for fixed end-events and a fixed range for w . Accordingly the optical rays satisfy Principle B as well as Principle A .

Traditionally, a principle of the form B has been preferred to one of form A . It has the advantage that there is no side-condition. But in the geometrical optics of a dispersive medium, the preference must be given to A , because the function $\omega(x, p)$ is to be regarded as given, whereas, to get L , we have to solve (43) for the p 's, and that may prove very difficult in practice.

Let us examine (43) with ω as in (23), so that [cf. (25) and (29)] we are required to solve for the p 's the four equations ¹

$$z^i = \frac{\partial \omega}{\partial p_i} = \bar{g}^{ij}p_j - nn'(p_k V^k)^2 V^i. \quad (55)$$

From this we get, in the notation (27),

$$p_i = \bar{g}_{ij}[z^j + nn'(p_k V^k)^2 V^j], \quad (56)$$

and hence

$$\begin{aligned} p_i z^i &= \bar{g}_{ij} z^i z^j + nn'(p_k V^k)^2 \bar{g}_{ij} z^i V^j, \\ \omega &= \frac{1}{2} \bar{g}^{ij} p_i p_j = \frac{1}{2} \bar{g}_{ij} z^i z^j + nn'(p_k V^k)^2 \bar{g}_{ij} z^i V^j - \frac{1}{2} n'^2 (p_k V^k)^4. \end{aligned} \quad (57)$$

In this last formula we have reduced the last term by noting that

$$\bar{g}_{ij} = g_{ij} + (1 - n^{-2}) V_i V_j, \quad (58)$$

so that

$$\bar{g}_{ij} V^i V^j = -n^{-2}. \quad (59)$$

By (57) we have

$$L(x, z) = p_i z^i - \omega = \frac{1}{2} \bar{g}_{ij} z^i z^j + \frac{1}{2} n'^2 (p_k V^k)^4. \quad (60)$$

¹ Do not confuse the meanings of the prime on n' and the prime on $x^{i'}$!

But we have not yet expressed L , as required, in terms of (x, z) , because the p 's are present, both explicitly and hidden in \bar{g}_{ij} and n' . To remove them, we note that

$$\bar{g}_{ij}V^iz^j = n^{-2}V_iz^i, \quad (61)$$

so that multiplication of (56) by V^i leads to the equation

$$nn'(p_iV^i)^2 + n^2p_iV^i = V_iz^i. \quad (62)$$

Since n is supposed given as a function of frequency, or equivalently of p_iV^i , (62) may be regarded as an equation to determine p_iV^i as a function of V_iz^i , so that we may write

$$p_iV^i = \psi(V_iz^i). \quad (63)$$

On substituting this in (60), including substitution in \bar{g}_{ij} and n' , we get $L(x, z)$ expressed as a function of (x, z) ¹.

Since dispersion is usually small in practice, it is natural to seek an approximation based on the smallness of n' . However this approximation is a delicate matter, and will not be pursued here; to find the rays in any actual case, one should not bother about a variational principle, but use the equations (41) for the rays. But although we cannot approximate for small n' , we can set $n' = 0$ (so that n becomes a function of position in space-time). This is the case of a *non-dispersive medium*, and for it the theory simplifies in a pleasant way.

For a non-dispersive medium, (60) gives

$$L(x, z) = \frac{1}{2}\bar{g}_{ij}z^iz^j, \quad (64)$$

with \bar{g}_{ij} as in (58). This is in the required form. By (57) we see that $\omega = L$, and so we must have $L = 0$ on a ray. Writing x' for z , Principle *B* now reads

$$\delta \int \bar{g}_{ij}x^{i'}x^{j'}dw = 0, \quad (65)$$

but of these extremals we are to take only those for which

$$\bar{g}_{ij}x^{i'}x^{j'} = 0. \quad (66)$$

Thus we have the remarkable result ²: *In a non-dispersive medium the rays are null geodesics with respect to the modified metric tensor*

$$\bar{g}_{ij} = g_{ij} + \left(1 - \frac{1}{n^2}\right)V_iV_j, \quad (67)$$

¹ We may check that (49) is satisfied by the function $L(x, z)$ so obtained.

² Cf. BALAZS [1955], PHAM MAU QUAN [1957a].

where n is the refractive index and V^i the 4-velocity of the medium.

We now pass to the third variational principle, which is not applicable when $\omega(x, p)$ is homogeneous in the p 's, as it is for a non-dispersive medium; accordingly we suppose the medium dispersive. We start with a formal calculation which leads from a given medium-equation

$$\omega(x, p) = 0 \quad (68)$$

to a function $F(x, z)$ of the x 's and four other variables z^i . Write down the equations

$$z^i = \theta \frac{\partial \omega}{\partial p_i}. \quad (69)$$

Solving for the p 's, we get

$$p_i = p_i\left(x, \frac{z}{\theta}\right). \quad (70)$$

Substitute this in (68) and solve for θ , obtaining

$$\theta = \theta(x, z); \quad (71)$$

θ is necessarily homogeneous of degree unity in the z 's. Substitute for θ in (70) and get p_i as a function of the x 's and the z 's, homogeneous of degree zero in the latter. Finally define the function $F(x, z)$ by ¹

$$F(x, z) = p_i z^i; \quad (72)$$

it is homogeneous of degree unity in the z 's. Observe (and this is important) that if we give any values to the x 's and the z 's, the values of the p 's given by (69) and (71) necessarily satisfy (68).

Now take any curve $x^i = x^i(w)$ and write $dx^i/dw = x^{i'}$. If we define the p 's by

$$\frac{\partial \omega}{\partial p_i} = \frac{x^{i'}}{\theta(x, x')}, \quad (73)$$

we know that (68) is satisfied, and by (72)

$$\int F(x, x') dw = \int p_i dx^i. \quad (74)$$

¹ It is here that the method breaks down if ω is homogeneous in the p 's, since then $\omega = 0$ implies $F = 0$. Although the variational principle C, as in (75), is not available for a non-dispersive medium in general, it is available in modified form for any medium in the statical case; cf. (102). For some more detailed calculations for a dispersive medium, see SYNGE [1956b], p. 47.

Accordingly Principle *A*, as in (40), leads to

$$\text{Principle } C: \quad \delta \int F(x, x') dw = 0, \quad (75)$$

for variation with fixed end events, without any side condition.

Note the difference between Principles *B* and *C*: (i) *B* applies to non-dispersive media, whereas *C* does not, and (ii) in *B* the parameter has fixed end values, whereas in *C* it is free on account of the homogeneity of *F*.

The classical principle of Fermat reads

$$\delta \int n d\sigma = 0, \quad (76)$$

where $d\sigma$ is an element in Euclidean 3-space. If we choose, as we may, $w = s$ in (75), we get

$$\delta \int F ds = 0, \quad (77)$$

which resembles (76) formally. But the analogy is not good, because $d\sigma$ and ds mean quite different things, and *F* is not the refractive index. The true analogue of Fermat's principle is given in the next section.

§ 4. GEOMETRICAL OPTICS IN A STATIC UNIVERSE

Consider a static universe with metric form

$$\Phi = g_{\alpha\beta} dx^\alpha dx^\beta + g_{44} (dx^4)^2, \quad (78)$$

where the *g*'s are independent of x^4 . In this universe we have a transparent medium¹ with world-lines along the x^4 -lines; consequently its 4-velocity V^i satisfies

$$V^\alpha = 0, \quad g_{44}(V^4)^2 = -1, \quad V^4 = \sqrt{-g^{44}}. \quad (79)$$

The refractive index *n* is a function of frequency ν and position x^α , but is independent of x^4 .

From the preceding general theory we quote the medium-equation

$$\omega(x, p) = 0, \quad (80)$$

where

$$\omega(x, p) = \frac{1}{2}[g^{ij}p_i p_j - (n^2 - 1)(p_i V^i)^2], \quad (81)$$

and the ray-equations

$$\frac{dx^i}{dw} = \frac{\partial \omega}{\partial p_i}, \quad \frac{dp_i}{dw} = -\frac{\partial \omega}{\partial x^i}. \quad (82)$$

¹ In VII-§ 9 we were concerned with a vacuum.

The outstanding feature of the static case is that ω is independent of x^4 , and the ray-equations give, along each ray,

$$p_4 = \text{const.} \quad (83)$$

Now

$$h\nu = -p_i V^i = -p_4 V^4 = -p_4 \sqrt{-g^{44}}, \quad (84)$$

so that, as we go along a ray, we know how the frequency changes; it is proportional to V^4 , or $(-g^{44})^{\frac{1}{2}}$, or $(-g_{44})^{-\frac{1}{2}}$. Thus we recover the spectral-shift formula VII-(233).

By (79) we reduce (81) to

$$\omega(x, p) = \frac{1}{2}(g^{\alpha\beta} p_\alpha p_\beta - n^2 \chi^2), \quad (85)$$

where we have written

$$\chi = p_4 V^4 = -h\nu; \quad (86)$$

the prescription of the medium gives n as a function of χ and x^α , and so the term $n^2 \chi^2$ in (85) is a given function of these quantities. We have

$$\frac{\partial \chi}{\partial p_\alpha} = 0, \quad \frac{\partial \chi}{\partial p_4} = V^4 = \sqrt{-g^{44}}, \quad \frac{\partial \chi}{\partial x^\alpha} = p_4 (\sqrt{-g^{44}})_{,\alpha}. \quad (87)$$

The ray-equations (82) now read explicitly

$$\begin{aligned} \frac{dx^\alpha}{dw} &= g^{\alpha\beta} p_\beta, & \frac{dx^4}{dw} &= -n\chi \frac{\partial}{\partial \chi} (n\chi) \cdot \sqrt{-g^{44}}, \\ \frac{dp_\alpha}{dw} &= -\frac{1}{2} g^{\beta\gamma}_{,\alpha} p_\beta p_\gamma + n\chi \frac{\partial}{\partial \chi} (n\chi) \cdot p_4 (\sqrt{-g^{44}})_{,\alpha}, \end{aligned} \quad (88)$$

together with $p_4 = \text{const.}$ The speed v of a ray relative to the medium is given by

$$v^2 = \frac{g_{\alpha\beta} dx^\alpha dx^\beta}{-g_{44} (dx^4)^2}, \quad (89)$$

where dx^i is a displacement along the ray. By (88), with use of (80) and (85),

$$g_{\alpha\beta} \frac{dx^\alpha}{dw} \frac{dx^\beta}{dw} = g^{\alpha\beta} p_\alpha p_\beta = n^2 \chi^2, \quad (90)$$

and hence

$$\frac{1}{v^2} = \left[\frac{\partial}{\partial \chi} (n\chi) \right]^2. \quad (91)$$

Thus we have

$$\frac{1}{v} = \left| \frac{\partial}{\partial \chi} (n\chi) \right| = \left| \frac{\partial}{\partial \nu} (n\nu) \right| = \left| \frac{\partial}{\partial \nu} \left(\frac{\nu}{u} \right) \right|, \quad (92)$$

where $u (= n^{-1})$ is the wave-speed. This is recognized as precisely the classical formula for the reciprocal of group-speed ¹.

If the medium is non-dispersive, the ray-equations simplify a little because $\partial n / \partial \chi = 0$. Also, as at (65), the rays may be treated as null geodesics for the modified metric tensor

$$\bar{g}_{ij} = g_{ij} + \left(1 - \frac{1}{n^2} \right) V_i V_j, \quad (93)$$

which now reduces to

$$\bar{g}_{\alpha\beta} = g_{\alpha\beta}, \quad \bar{g}_{\alpha 4} = 0, \quad \bar{g}_{44} = n^{-2} g_{44}. \quad (94)$$

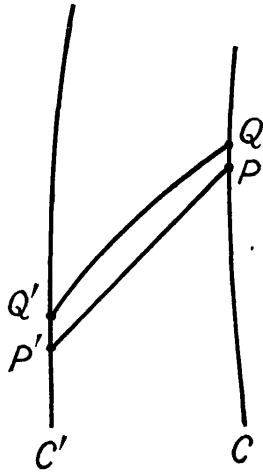


Fig. 6—Variational principle *A* in the static case

As regards variational principles in the statical case, for a medium which is in general dispersive, the simplest is Principle *A* of (40), which can be modified so that the time-coordinate x^4 does not appear. Fig. 6 shows two x^4 -lines, C' and C , and a ray $P'P$ joining them. We compare this ray with an adjacent curve $Q'Q$ joining C' and C , but in general with new end-events. We have already seen that p_4 is constant along $P'P$. We now assign on $Q'Q$ that same value of p_4 , and give to the remaining components p_α any values consistent with $\omega(x, p) = 0$, which relation, we recall, does not contain x^4 . Passing from $P'P$ to $Q'Q$, we have

$$\begin{aligned} \delta \int p_\alpha dx^\alpha &= \int (\delta p_\alpha dx^\alpha + p_\alpha \delta dx^\alpha) \\ &= \int (\delta p_\alpha dx^\alpha - \delta x^\alpha dp_\alpha) \\ &= \int \left(\frac{\partial \omega}{\partial p_\alpha} \delta p_\alpha + \frac{\partial \omega}{\partial x^\alpha} \delta x^\alpha \right) dw \\ &= \int \left(\delta \omega - \frac{\partial \omega}{\partial p_4} \delta p_4 - \frac{\partial \omega}{\partial x^4} \delta x^4 \right) dw. \end{aligned} \quad (95)$$

¹ The identity of ray-speed and group-speed was already established more generally in § 2.

But $\delta\omega = 0$, $\delta p_4 = 0$, $\partial\omega/\partial x^4 = 0$, and so we get the variational principle

$$\delta \int p_\alpha dx^\alpha = 0, \quad \omega(x, p) = 0, \quad (96)$$

with p_4 unvaried, as stated above, and with the end-events free to slide along the x^4 -lines. This is a truly static principle.

We may regard (96) as a Principle *A*, involving x^α and p_α ; x^4 is absent and p_4 is to be regarded merely as a fixed quantity. Then, by the same mathematical technique as before, but in a lower dimensionality, we can derive static Principles *B* and *C*.

Let us derive the static Principle *C*. As in (69), we have to solve, with ω as in (85),

$$z^\alpha = \theta \frac{\partial\omega}{\partial p_\alpha} = \theta g^{\alpha\beta} p_\beta. \quad (97)$$

Hence

$$p_\alpha = g_{\alpha\beta} z^\beta / \theta, \quad (98)$$

and when we substitute this in $\omega = 0$, we get

$$\theta^2 = \frac{g_{\alpha\beta} z^\alpha z^\beta}{n^2 \chi^2}. \quad (99)$$

Then (72) gives

$$F(x, z) = p_\alpha z^\alpha = \frac{g_{\alpha\beta} z^\alpha z^\beta}{\theta} = n\chi \sqrt{g_{\alpha\beta} z^\alpha z^\beta}. \quad (100)$$

We are to use this function in Principle *C* as in (75). Now

$$\chi = p_4 V^4 = p_4 \sqrt{-g^{44}}, \quad (101)$$

and since p_4 is fixed, we can drop it. Thus we get the variational principle (Fermat type) ¹

$$\delta \int n \sqrt{-g^{44} g_{\alpha\beta} x^{\alpha'} x^{\beta'}} dw = 0, \quad (102)$$

the end-points being fixed and w being an arbitrary parameter.

If we choose $w = \sigma =$ spatial distance, so that

$$d\sigma^2 = g_{\alpha\beta} dx^\alpha dx^\beta, \quad (103)$$

(102) becomes

$$\delta \int n \sqrt{-g^{44}} d\sigma = 0, \quad (104)$$

¹ For simplicity we have confined ourselves to the statical case, but the general plan can be applied to the stationary case in which $g_{\alpha 4} \neq 0$ and all quantities are independent of x^4 . For Fermat's principle in the stationary case, see LEVI-CIVITÀ [1918c], [1927], SYNGE [1925].

which comes very close to Fermat's principle in the classical form (76), the index n being modified by the factor shown.

In the above theory, it does not matter whether the medium is dispersive or not, because the assignment of p_4 has confined the systems under consideration to one frequency only. But if we desire to obtain a principle of stationary time, we are forced to take the medium non-dispersive. Then (92) tells us that the ray-speed v equals the wave-speed u , which equals n^{-1} , so that

$$v = \frac{d\sigma}{\sqrt{-g_{44}}dx^4} = n^{-1}, \quad n\sqrt{-g_{44}}d\sigma = dx^4, \quad (105)$$

and (104) becomes the *principle of stationary time*,

$$\delta \int dx^4 = 0. \quad (106)$$

§ 5. ASTRONOMICAL OBSERVATIONS

Newtonian theory continues to be used with great success in celestial mechanics, but there are two skeletons in the cupboard. First, although celestial mechanics involves no optical ideas, astronomical observation is optical and it is impossible to fit optics into the Newtonian scheme. When the astronomer asks himself, not what the phenomena *are*, but how they should be *seen* by him, he is compelled to use different 'ethers' for different problems. Secondly, the instantaneous propagation of gravity is an idea contrary to the spirit of modern physics, but it is impossible to fit a finite speed of propagation into Newtonian gravitation. However, it is doubtful whether any modern astronomer would wish to defend the Newtonian theory as a correct representation of nature; he is more likely to regard it as a very successful compromise, saved from open conflict with reality by the steadiness of the major gravitational fields, the weakness of the variable ones, and the smallness of the relative velocities of celestial bodies, the planets in particular.¹

In comparison with Newtonian theory, relativity is clumsy and does not offer a clear picture of the problems of celestial mechanics. But for all its clumsiness it is honest. The relativistic cupboard is untidy, but there are no skeletons in it — at least we hope not. We are not able

¹ The need for closer connection between practical astronomy and relativity was stressed, and some of the formulae which follow were presented, in lectures in Milan in 1959; cf. SYNGE [1960e].

to calculate the Riemann tensor throughout the solar system, but, assuming it known, we can discuss celestial mechanics on the basis of the geodesic hypothesis. Moreover, there is no embarrassment about 'ethers'; the problem of astronomical observation is a problem in the geometry of null geodesics.

To illustrate by a particular example, if the orbits of Mars and the earth are given, the problem of astronomical observation is that of predicting how the terrestrial observer should direct his telescope in order to keep Mars on the cross-wires, and how spectral lines emitted by Mars are shifted by its motion. But we can at once pass on to the general problem of astronomical observation, involving a source and an observer, with world-lines not necessarily geodesic. Then the whole matter is contained in the geometry of a two-dimensional strip in space-time (Fig. 7), built up of null geodesics I and terminated by two timelike world-lines, C_1 for the observer and C_2 for the source.

This appears simple, but the calculations are necessarily somewhat complicated, and it is essential to control them by some central idea. For this we shall use the world-function Ω of Chap. II, although the equation of geodesic deviation I-(130) might be employed instead. One should remember that Ω is conceptually very simple, being (to within a factor $\pm \frac{1}{2}$) the square of the geodesic 'distance' between two events, regarded as a function of their eight coordinates.

To be realistic, we should place the observer, and perhaps the source also, in a refracting medium. But this would make the problem too complicated, and we shall assume vacuum conditions throughout, so that the fundamental properties of a photon passing from source to observer are (i) its world-line is a null geodesic, and (ii) its 4-momentum p^i is tangent to the world-line and undergoes parallel transport along it.

The astronomer measures direction with a telescope and frequency with a spectrometer. These observations are equivalent to measuring p^i for a photon. To explain this, we refer to III-§ 6 where the measurement of direction was discussed. If $\lambda_{(a)}^i$ is an orthonormal tetrad on C_1 with $\lambda_{(4)}^i$ tangent to it, then the components of the photon's momentum

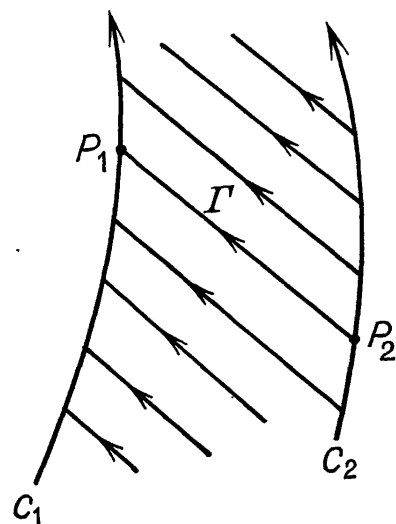


Fig. 7 – The problem of astronomical observation

are

$$p^{(\alpha)} = p_{(\alpha)} = p_i \lambda_{(\alpha)}^i, \quad (107)$$

and their ratios are the direction ratios of the telescope which catches the photon. Further, the energy of the photon is

$$h\nu = p^{(4)} = -p_{(4)} = -p_i \lambda_{(4)}^i, \quad (108)$$

so that the measurement of the frequency ν gives $p^{(4)}$. Since p^i is null, we have

$$p^{(\alpha)} p^{(\alpha)} = (p^{(4)})^2. \quad (109)$$

Thus, although the astronomer measures only *three* quantities (two angles and the frequency), he determines all *four* quantities $p^{(a)}$, and hence p^i (in any chosen coordinate system) since ¹

$$p^i = p^{(a)} \lambda_{(a)}^i. \quad (110)$$

Noting then that p^i are observable quantities, we consider the observer with world-line C_1 (Fig. 7). All optical information reaching him at the event P_1 comes from events on the null cone with vertex P_1 , drawn into the past ². Let C_2 be the world-line of a source emitting photons of frequency ν_0 relative to C_2 . We shall suppose ν_0 constant, i.e. it corresponds to the emission of some definite spectral line.

The histories of all photons passing from source to observer form a 2-space composed of null geodesics. Let v be a parameter which is constant on each of these, with $v = s =$ observer's time on C_1 . We write

$$V^{i_1} = \left(\frac{dx^i}{dv} \right)_{P_1}, \quad V^{i_2} = \left(\frac{dx^i}{dv} \right)_{P_2}, \quad (111)$$

and denote the corresponding 4-velocities by A^{i_1} and A^{i_2} ; then

$$A^{i_1} = V^{i_1}, \quad A^{i_2} = \frac{V^{i_2}}{\sqrt{-V_{j_2} V^{j_2}}}. \quad (112)$$

Let p^i be the 4-momentum of a photon passing from P_2 to P_1 . We have ³

$$h\nu_0 = -p_{i_2} A^{i_2}. \quad (113)$$

¹ We recall that the labels are raised and lowered by means of $\eta^{(ab)} = \eta_{(ab)} = \text{diag}(1, 1, 1, -1)$.

² For a terrestrial observer, about half this null cone is blocked by the solid earth; naturally we are interested only in the part of the cone which is not thus blocked.

³ Throughout this work the secondary numerical suffixes refer to P_1 and P_2 . As in Chap II, we denote partial and covariant derivatives of Ω by subscripts without any other sign.

But in terms of the partial derivatives of the world-function $\Omega(P_1P_2)$, the 4-momenta of the photon at P_1 and P_2 are given by

$$p_{i_1} = \chi \Omega_{i_1}, \quad p_{i_2} = -\chi \Omega_{i_2}, \quad (114)$$

where χ is constant along P_1P_2 ; by (113) its value is

$$\chi = \frac{h\nu_0}{\Omega_{i_2} A^{i_2}}, \quad (115)$$

and so, by (114),

$$p_{i_1} = \frac{h\nu_0 \Omega_{i_1}}{\Omega_{j_2} A^{j_2}}. \quad (116)$$

This formula contains the whole story of optical observations of the type considered. If the world-function is known, together with the events of emission and reception and the 4-velocity of the source at emission, then (116) gives the 4-momentum of the photon when received by the observer.

If, for a weak gravitational field, we use coordinates such that, as in VII-(240),

$$g_{ij} = \eta_{ij} + \gamma_{ij}, \quad (117)$$

with γ_{ij} small, we have in VII-(250) formulae for the partial derivatives of Ω . These may be substituted in (116) to solve the problem of astronomical observation as presented here. For a static field, in particular, we have VII-(253), while for the solar field we can calculate the required derivatives from VII-(256). Indeed, the question of spectral shift was already dealt with in VII-(270). We shall not attempt here to fill in details with regard to observed direction.

The use of (117) seems adequate in dealing with phenomena in the solar system, but this coordinate system becomes unreliable at great distances, and we shall not use it in the discussion of stellar aberration in the next section.

§ 6. STELLAR ABERRATION

The pattern formed by the stars is observed to undergo a systematic change, with period one year, which is described by saying that each star describes a small ellipse on the celestial sphere, the ellipse becoming a circle at the pole of the ecliptic and a straight line on the ecliptic. The radius of the circle at the pole is v/c radians, where v is the

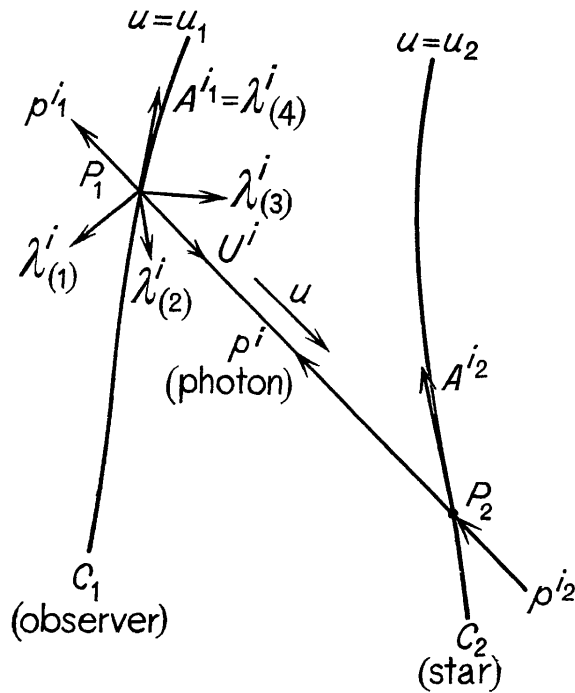


Fig. 8 – Stellar aberration

orbital speed of the earth about the sun and c the speed of light; this angle is $20.''5$, and it is called the *constant of aberration*. The length of the straight line on the ecliptic is twice this constant. It is further reported that there is a diurnal aberration, depending on the observer's latitude and ranging from $0.''31$ at the equator to zero at the pole. To explain these facts, the astronomer uses an ether in which the sun is fixed. We seek an explanation in terms of curved space-time ¹.

Fig. 8 is an elaboration of Fig. 7. We start with a star at finite distance with world-line C_2 , and on the null geodesics running back from C_1 to C_2 we assign a special parameter u , running between fixed end values, u_1 on C_1 and u_2 on C_2 , with $u_1 < u_2$ (the photon passes in the sense of u decreasing). We write

$$U^i = dx^i/du, \quad k^{-1} = u_2 - u_1. \quad (118)$$

Then we have by II-(17)

$$k\Omega_{i_1} = -U_{i_1}, \quad k\Omega_{i_2} = U_{i_2}, \quad (119)$$

and as we go along C_1 (here $D = \delta/\delta s$)

$$DU_{i_1} = -k(\Omega_{i_1 j_1} A^{j_1} + \Omega_{i_1 j_2} V^{j_2}). \quad (120)$$

We shall now remove the star to infinity along the null geodesic P_1P_2 , keeping the same parameter u . This means that $(u_2 - u_1)$ tends to infinity, or, equivalently, k tends to zero. We assume the field weak everywhere and neglect terms quadratic in the Riemann tensor, so that the second derivatives of Ω are as given in II-(95), with the O_2 term omitted. Noting that in the Schwarzschild field the Riemann tensor falls off as $1/r^3$, we recognize that for the greater part of its history the photon passes through space-time that is

¹ Cf. MAST and STRATHDEE [1959]; the method used here differs in some respects from theirs.

very nearly flat. For mathematical convenience, we idealize this by introducing the cut-off shown in Fig. 6 of I-§ 6. In fact, we assume $R_{ijkl} = 0$ except for

$$u_1 \leq u \leq \bar{u}_1, \quad \bar{u}_2 \leq u \leq u_2. \quad (121)$$

As we mentally remove the star to infinity, we keep $(\bar{u}_1 - u_1)$ and $(u_2 - \bar{u}_2)$ finite. It is easy to see then that II-(95) gives

$$\lim_{k \rightarrow 0} k\Omega_{i_1 j_1} = -W_{i_1 j_1}, \quad \lim_{k \rightarrow 0} k\Omega_{i_1 j_2} = 0, \quad (122)$$

where, by II-(69),

$$\begin{aligned} W_{i_1 j_1} &= -\frac{3}{2} \int_{u_1}^{\bar{u}_1} g_{i_1 a} g_{j_1 b} S^{abpq} U_p U_q du \\ &= \int_{u_1}^{\bar{u}_1} g_{i_1 a} g_{j_1 b} R^{apbq} U_p U_q du. \end{aligned} \quad (123)$$

We are to substitute from (122) in (120), but the time has come to simplify the notation by dropping the secondary subscript 1. For an infinitely distant star, we have then ¹

$$DU_i = W_{ij} A^j, \quad (124)$$

where A^j is the 4-velocity of the observer and

$$W_{ij} = W_{ji} = \int_{u_1}^{\bar{u}_1} g_{ia'} g_{jb'} R^{a'p'b'q'} U_{p'} U_{q'} du. \quad (125)$$

This tensor W_{ij} may be called the *aberration tensor* since it controls stellar aberration. Its value depends on the event P on the observer's world-line and on the direction of the star, but not on the 4-velocity of the observer, nor on his 4-acceleration. We note that, since U^i undergoes parallel transport on the null geodesic,

$$W_{ij} U^j = 0. \quad (126)$$

There is a rather subtle point about the special parameter u on the null geodesics joining C_1 and C_2 . Before we went to the limit of infinite distance, we might have chosen u on one of the null geodesics as any one of the linearly related special parameters, but, having so chosen it, it would have been fixed on the other null geodesics by the requirement that it should take constant values on C_1 and C_2 . This

¹ This formula may be checked against I-(157), noting the different meanings of D . The primes in (125) refer to a current event on the null geodesic; $g_{ia'}$ is the parallel propagator of II-(71).

restriction is not removed by taking the star to a very great distance, and we are entitled to make an arbitrary choice of u on only *one* of the null geodesics. However, in applying (124) at any *one* event P , we are entitled to normalize u , and we shall do this by the following equivalent demands:

$$U_i A^i = 1, \quad U_{(4)} = 1, \quad U^{(4)} = -1, \quad (127)$$

$\lambda_{(a)}^i$ being an orthonormal tetrad on C_1 with $\lambda_{(4)}^i$ tangent to it. We must be careful to use (127) only *after* differentiating.

Now

$$U_{(\alpha)} = U_i \lambda_{(\alpha)}^i \quad (128)$$

are the direction ratios, looking outward, of the telescope relative to the triad $\lambda_{(\alpha)}^i$, and so the direction cosines of the telescope are

$$l_{(\alpha)} = \frac{U_{(\alpha)}}{U_{(4)}} = \frac{U_i \lambda_{(\alpha)}^i}{U_j A^j}. \quad (129)$$

For the rates of change of these direction cosines, we have, with the aid of (127),

$$Dl_{(\alpha)} = DU_{(\alpha)} - U_{(\alpha)} D(U_j A^j). \quad (130)$$

So far the triad $\lambda_{(\alpha)}^i$ is arbitrary. We now make it a Fermi triad, so that, as in I-(84),

$$D\lambda_{(\alpha)}^i = A^i \kappa_j \lambda_{(\alpha)}^j, \quad (131)$$

where κ_j is the first curvature vector of C_1 ($\kappa_j = DA_j$). Then

$$\begin{aligned} DU_{(\alpha)} &= D(U_i \lambda_{(\alpha)}^i) = \kappa_i \lambda_{(\alpha)}^i + W_{ij} \lambda_{(\alpha)}^i A^j = \kappa_{(\alpha)} + W_{(\alpha 4)}, \\ D(U_j A^j) &= U_j \kappa^j + W_{jk} A^j A^k = U_{(\alpha)} \kappa_{(\alpha)} + W_{(44)}, \end{aligned} \quad (132)$$

the subscripts in parentheses indicating components on the tetrad $\lambda_{(a)}^i$ ($\lambda_{(4)}^i = A^i$). Then (130) gives

$$Dl_{(\alpha)} = \kappa_{(\alpha)} + W_{(\alpha 4)} - U_{(\alpha)} U_{(\beta)} \kappa_{(\beta)} - U_{(\alpha)} W_{(44)}. \quad (133)$$

By (126) and (127),

$$W_{(\alpha\beta)} U_{(\beta)} - W_{(\alpha 4)} = 0, \quad W_{(4\beta)} U_{(\beta)} - W_{(44)} = 0, \quad (134)$$

so that

$$W_{(\alpha 4)} = W_{(\alpha\beta)} U_{(\beta)}, \quad W_{(44)} = W_{(\alpha\beta)} U_{(\alpha)} U_{(\beta)}. \quad (135)$$

Since we have

$$U_{(\alpha)} = l_{(\alpha)}, \quad (136)$$

(133) may be written ¹

$$Dl_{(\alpha)} = P_{(\alpha\beta)}(\kappa_{(\beta)} + W_{(\beta\gamma)}l_{(\gamma)}), \quad (137)$$

where $P_{(\alpha\beta)}$ is the three-dimensional projection operator

$$P_{(\alpha\beta)} = \delta_{\alpha\beta} - l_{(\alpha)}l_{(\beta)}. \quad (138)$$

Let us see what (137) means for the astronomer observing a star. He takes the triad $\lambda_{(\alpha)}^i$ as coordinate axes and draws a celestial sphere of unit radius (Fig. 9). The star appears as a point with coordinates $l_{(\alpha)}$.

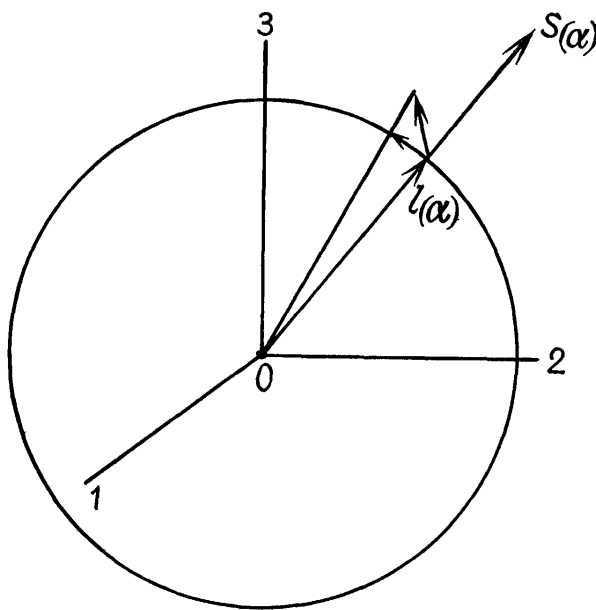


Fig. 9 – The celestial sphere

As time passes, the point moves: that is the phenomenon of stellar aberration. By (137) we see that, in a small time ds , the point $l_{(\alpha)}$ receives a displacement which can be divided into two parts. First, the point is moved *off* the sphere by a displacement

$$(\kappa_{(\alpha)} + W_{(\alpha\gamma)}l_{(\gamma)})ds. \quad (139)$$

Secondly, it is brought back on to the sphere by a radial displacement. The displacement (139) consists of a part $(\kappa_{(\alpha)}ds)$, the same for all

¹ It is of course understood that the parameter u is normalized as in (127). It is easy to see that

$$W_{(\beta\gamma)}l_{(\gamma)} = l_{(\gamma)} \int R_{(\beta 4 \gamma 4)} du - l_{(\rho)} l_{(\gamma)} \int R_{(\beta \rho \gamma 4)} du, \quad (137a)$$

where the range of integration is as in (125) and the R -terms are components of the Riemann tensor on an orthonormal tetrad obtained from $\lambda_{(\alpha)}^i$ by parallel transport on P_1P_2 (the primes on the subscripts are omitted for simplicity)

stars, and a second part which depends on the particular star under observation.

Does (137) tell us that stellar aberration is small? For a terrestrial observer, $\kappa_{(\alpha)}$ has a very simple meaning. It is a vector directed upwards with a magnitude equal to the usual g (the so-called acceleration due to gravity; cf. the falling apple of III-§ 9); the numerical value is $3.3 \times 10^{-8} \text{ sec}^{-1}$, and this indeed seems small. But if we integrate over a year ($= 3.2 \times 10^7 \text{ sec}$), it seems that the κ -term in (137) may make a contribution of the order of unity! The physical fact that stellar aberration is small tells us that the two terms in (137) must largely cancel one another.

Given an event P , a 4-velocity A^i , and the direction $l_{(\alpha)}$ of a star, we can choose the *acceleration* of the observer so as to make aberration vanish. We have merely to give to his world-line a first curvature

$$\bar{\kappa}_{(\alpha)} = -W_{(\alpha\gamma)}l_{(\gamma)}. \quad (140)$$

In fact, for a given star, this equation defines throughout space-time a complex of *curves of no aberration*, one passing through each event in each direction. With $\bar{\kappa}_{(\alpha)}$ so defined, we may write (137) as

$$Dl_{(\alpha)} = P_{(\alpha\beta)}(\kappa_{(\beta)} - \bar{\kappa}_{(\beta)}). \quad (141)$$

That may appear to be an empty notational gesture, but it is not entirely so. It would certainly not be empty if we knew the curves of no aberration, and in the case of a stationary universe we know, not the whole complex, but at least a congruence belonging to the complex. For in a stationary universe we have $g_{ij,4} = 0$, and space-time admits a group of motions along the x^4 -lines. It is evident that for an observer who has an x^4 -line for world-line there can be no stellar aberration, and so the congruence of x^4 -lines are curves of no aberration.

In the case of an actual terrestrial observer, the field is not stationary, and we must abandon the attempt to make a complete discussion of stellar aberration for him. Let us idealize by putting the observer on a massless particle in a stationary field, possibly the field of the sun. It is not necessary to make his world-line a geodesic, but we shall do so for simplicity, putting $\kappa_{(\alpha)} = 0$; in the idealization of the terrestrial observer, this geodesic assumption is reasonable. Then (141) gives

$$Dl_{(\alpha)} = -P_{(\alpha\beta)}\bar{\kappa}_{(\beta)}. \quad (142)$$

Here $\bar{\kappa}_{(\alpha)}$ are the components on the observer's triad of the first curvature vector (say $\bar{\kappa}_i$) of the curve of no aberration A tangent to

the observer's world-line C ; in fact,

$$\bar{\kappa}_{(\alpha)} = \bar{\kappa}_i \lambda_{(\alpha)}^i. \quad (143)$$

Fig. 10 shows the curves C and A and also the x^4 -line T .

We do not know the value of $\bar{\kappa}^i$, but we can easily calculate the first curvature vector of T (say $\bar{\kappa}'^i$) from the general formula

$$\kappa^i = \frac{d^2 x^i}{ds^2} + \Gamma_{jk}^i \frac{dx^j}{ds} \frac{dx^k}{ds}; \quad (144)$$

it gives

$$\bar{\kappa}'^i = -\frac{\Gamma_{44}^i}{g_{44}}, \quad \bar{\kappa}'_i = (\log \sqrt{-g_{44}})_{,i} \quad (145)$$

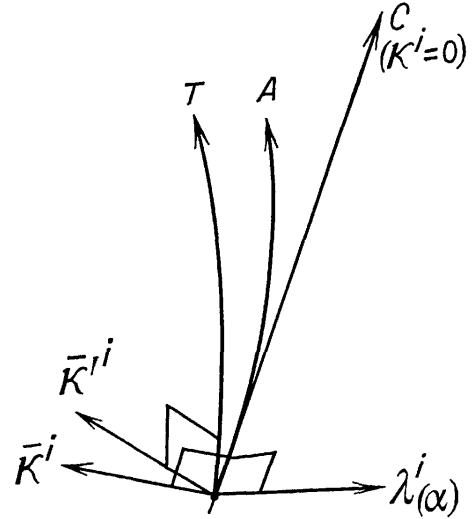


Fig. 10 – The observer's world-line C , the curve of no aberration A tangent to C , and the x^4 -line T

To make use of this knowledge, we must at this point make an approximation. The weakness of the field has already been assumed, and its stationary character; we now add the assumption that the observer is moving *slowly*, by which we mean that the direction of C is nearly the same as that of T . It is natural to suppose that, for two curves of no aberration which are nearly in the same direction, the first curvature vectors are nearly the same. Thus

$$\bar{\kappa}_{(\alpha)} = \bar{\kappa}_i \lambda_{(\alpha)}^i \approx \bar{\kappa}'_i \lambda_{(\alpha)}^i = (\log \sqrt{-g_{44}})_{,\beta} \lambda_{(\alpha)}^\beta. \quad (146)$$

Accepting this as a valid approximation, we are to substitute in (142), replacing $\bar{\kappa}_{(\beta)}$ (which depends on the direction of the star) by an expression which does not so depend. Understanding that $\bar{\kappa}_{(\beta)}$ now has the star-free value, we re-examine the displacement in the present case. We see that in time ds the point $l_{(\alpha)}$ is pulled off the sphere by a displacement $-\bar{\kappa}_{(\alpha)}ds$, *the same for all very distant stars*, and then pushed back again radially. But why bother to push it back? The aberrational motion of the star is described more simply by abandoning the unit vector $l_{(\alpha)}$ and using instead a vector $S_{(\alpha)}$ which starts with the direction of $l_{(\alpha)}$ and satisfies

$$DS_{(\alpha)} = -\bar{\kappa}_{(\alpha)}. \quad (147)$$

It will of course keep the same direction as $l_{(\alpha)}$.

We have before us now the aberrational equation

$$DS_{(\alpha)} = -(\log \sqrt{-g_{44}})_{,\beta} \lambda_{(\alpha)}^\beta. \quad (148)$$

We recall that we are dealing only with the aberration of very (infinitely) distant stars, that the field around the observer is weak and stationary, that $S_{(\alpha)}$ points in the apparent direction of the star as viewed through a telescope, and that D indicates rate of change with respect to the observer's time. The field being weak, we understand that $g_{ij,k}$ are small. Since C is a geodesic, the Fermi transport of $\lambda_{(\alpha)}^i$ becomes parallel transport, and we may treat the λ 's in (148) as constants.

We might leave (148) as our final approximate result, but it is interesting to connect it with the classical explanation of aberration in terms of the velocity of the observer. Writing, as a more suggestive notation, $v^i = A^i = dx^i/ds$, $v_i = g_{ij}v^j$ for the observer's geodesic world-line C , we have the equations

$$\frac{dv_i}{ds} - \frac{1}{2}g_{jk,i}v^jv^k = 0, \quad (149)$$

and these give approximately

$$\frac{dv_\beta}{ds} + (\log \sqrt{-g_{44}})_{,\beta} = 0. \quad (150)$$

Hence the aberrational equation (148) may be written

$$\frac{dS_{(\alpha)}}{ds} = \frac{dv_\beta}{ds} \lambda_{(\alpha)}^\beta. \quad (151)$$

Denoting by Δ increments corresponding to any finite time, we have

$$\Delta S_{(\alpha)} = \Delta v_\beta \cdot \lambda_{(\alpha)}^\beta, \quad (152)$$

which is essentially the classical statement that the aberrational displacement is equal to the velocity of the observer.

Let us leave these rather unpleasant approximations and go back to (137), in which the only assumptions are that (i) the star is infinitely distant, and (ii) the field is weak, with a cut-off (but this last is really only a device for mathematical treatment). According to the classical explanation of stellar aberration in the case of very distant stars, an observer could eliminate it by remaining 'at rest'. We might invert the argument, and say that an observer is 'at rest' if he sees no aberration. What does relativity say to this? Can an observer so choose his world-line that he sees no aberration? Apparently not. If he is interested in

only one particular star, he can get rid of its aberration by following a curve of no aberration as in (140), but unless the right hand side of (140) happens to be independent of direction, other stars will be aberrated. There is an interesting question here: can we define *rest-curves* in space-time statistically by a criterion of minimizing a suitably defined mean aberration?

In dealing with aberration, we have used throughout a Fermi-transported reference triad, and it seems mathematically appropriate. The change to another triad is really rather trivial — the whole instantaneous pattern of stars is given a rigid rotation.

§ 7. DIFFERENTIAL CHRONOMETRY

In an interferometer, light from a source is offered two alternative paths, and interference patterns are produced by the two returning beams. Although there is no obvious time-measurement here, it is recognized that what matters in such a case is the difference between two times taken by the light following the two permitted paths, and therefore one is entitled to call an interferometer a *differential chronometer* (briefly, DC). The interferometer of Michelson and Morley was a DC, and what is to be discussed below might be regarded a generalization of their apparatus. But in view of modern developments in the accurate measurement of time (down to 10^{-10} sec) one should not regard the interference of light as an essential feature of a DC; there may be better techniques.

We shall be concerned with light signals (or other electromagnetic signals) which go round circuits, the time taken to complete the circuit (the *trip-time*) being measured on a clock carried by a source which forms the beginning and end of the circuit. Except for instantaneous reflection at mirrors, the light travels in vacuo, so that we are concerned with a geometrical problem involving null geodesics and the world-lines of the source and the mirrors. We assume that the geometry of space-time is assigned, i.e. it is not affected by the experiments performed. The purpose behind such experiments would be to determine the first curvature (4-acceleration) of the world-line of the source, its other two curvatures, and the curvature of space-time (i.e. the gravitational field).

We shall be careful not to spoil the argument by admitting the concept of rigid bodies, as is often done in discussions of the Michelson-Morley experiment.

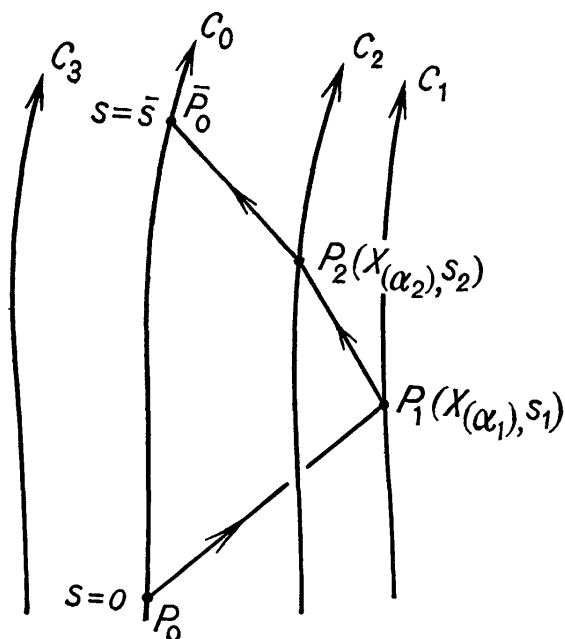


Fig. 11 — A tetrahedral differential chronometer in space-time

Fig. 11 is a space-time picture of a tetrahedral DC; C_0 is the world-line of the source, carrying a clock, and C_1, C_2, C_3 are the world-lines of three mirrors, idealized to mere points. To be realistic, we may think of C_0 as pertaining to a point fixed on the earth's surface, or moving about on the earth's surface in some specified way. In any case, we regard C_0 as assigned, while the other three world-lines are to be controlled according to our desires.

We see in Fig. 11 the history of a photon (signal) which goes from C_0 at P_0 to C_1 , then to C_2 , finally returning to C_0 at \bar{P}_0 . The symbol $[0120]$ will denote this circuit or the trip-time $P_0\bar{P}_0$. Symbols such as $[010]$, $[0230]$ have similar meanings.

Each world-line has three degrees of freedom, and so we have altogether nine degrees of freedom at our disposal. We shall use up six of these by controlling the mirrors so that

$$[010] = [020] = [030] = 2T \quad (153)$$

and

$$[0230320] = [0310130] = [0120210] = (4 + 2\sqrt{2})T, \quad (154)$$

where T is some arbitrary constant.

To understand these strange-looking conditions, we introduce an orthonormal tetrad $\lambda_{(a)}^i$ on C_0 , with $\lambda_{(4)}^i = A^i$ (the 4-velocity of C_0) and $\lambda_{(\alpha)}^i$ carried along by Fermi transport. Let $X^{(\alpha)}$ ($= X_{(\alpha)}$) be Fermi coordinates¹ relative to C_0 , as defined in II-§ 10. We can now draw pictures in the Fermi 3-space in which we use $X_{(\alpha)}$ as rectangular Cartesian coordinates, with the fourth Fermi coordinate $X^{(4)} (= s \text{ on } C_0)$ considered as a sort of Newtonian time. The history of each mirror is given by equations of the form $X_{(\alpha)} = X_{(\alpha)}(X^{(4)}) = X_{(\alpha)}(s)$. We must be a little careful in speaking of the 'distance' between two mirrors,

¹ Since in the DC light goes both ways, these are more convenient than the optical coordinates of II-§ 10.

because the term can be defined in different ways. But it is usually most convenient to use the *Fermi distance*, such that the square of the distance between C_1 and C_2 is, for fixed s on C_0 ,

$$r_{12}^2 = (X_{(\alpha_1)} - X_{(\alpha_2)})(X_{(\alpha_1)} - X_{(\alpha_2)}). \quad (155)$$

There is no danger in using the same names for the world-lines and for the corresponding points in the Fermi 3-space. The picture of the DC for any given value of s shows a tetrahedron as in Fig. 12. As s

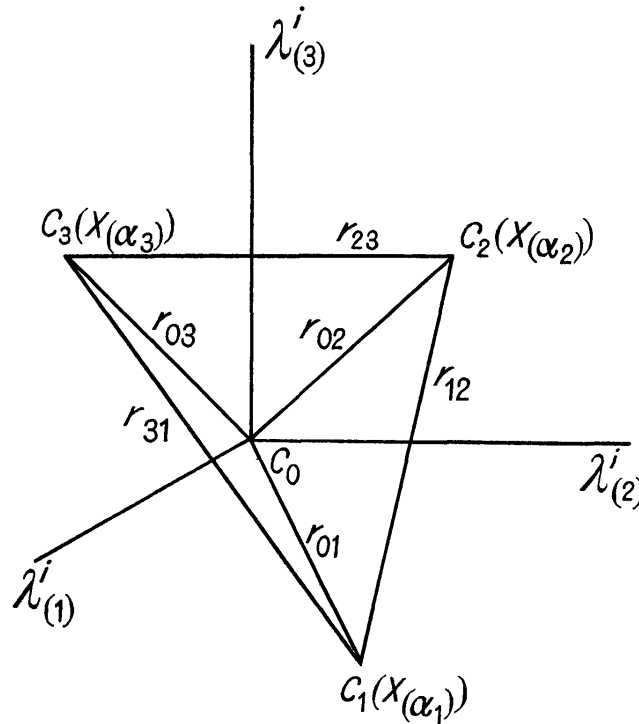


Fig. 12 – Tetrahedral DC in Fermi 3-space

changes, the points move, but C_0 remains permanently at the origin.

If space-time were flat, C_0 a geodesic, and the tetrahedron not rotating, then the conditions (153) and (154) would ensure that the tetrahedron would have the form shown in Fig. 13, with three mutually perpendicular edges of length T . Since the above conditions will be nearly satisfied in all cases of physical interest, the instantaneous picture of the tetrahedron in Fermi space will not differ much from this, but we must be prepared for small distortions.

In order to find out about the behaviour of the DC, we must face some heavy calculations, based on those of II-§ 14 and Fig. 16 of that section. The fundamental assumption for approximation is the smallness of the distances between the several world-lines. We add also the assumption that the curvature of C_0 is small and the Fermi

coordinates nearly constant. We shall in fact drop the term N_4 in II-(279) and treat M_3 and N_3 as O_4 , but we shall keep M_4 in order to explore the effect of gravitation, even though we know that the curvature of space-time is small. Thus we write for the world-function

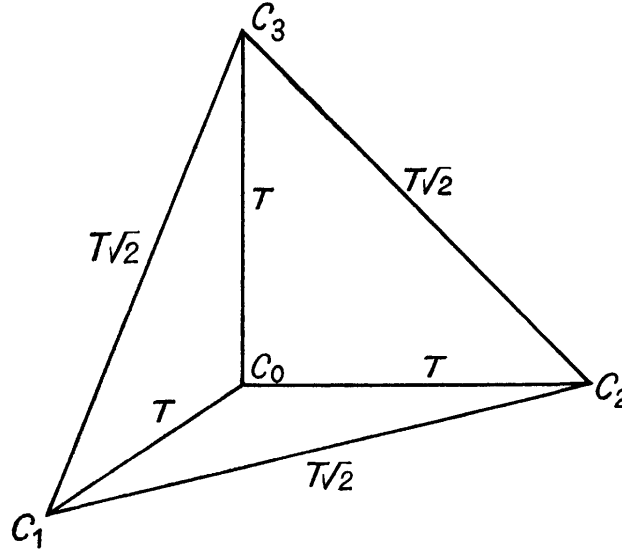


Fig. 13 – The standard tetrahedron

of any two events P_1, P_2 on C_1, C_2 respectively,

$$\Omega(P_1P_2) = M_2 + M_3 + N_3 + M_4 + O_5, \quad (156)$$

where the terms are as in II-(280) to II-(283). Now the circuit [0120] shown in Fig. 11 is composed of three null geodesics, and so

$$\Omega(P_0P_1) = 0, \quad \Omega(P_1P_2) = 0, \quad \Omega(P_2\bar{P}_0) = 0. \quad (157)$$

Our object is to calculate the trip-time [0120] and others, such as [010], but this can all be done by concentrating on the second equation in (157), using it to find $(s_2 - s_1)$, and then deriving the other needed quantities by playing with the numerical subscripts.

Using II-(280), we get from (156) and the second of (157)

$$(s_2 - s_1)^2 = r_{12}^2 + 2M_3 + 2N_3 + 2M_4 + O_5. \quad (158)$$

Hence, with an unwritten error term O_4 and with $D = d/ds$,

$$s_2 - s_1 = r_{12} + s_1 D r_{12} - (X_{(\alpha_1)} - X_{(\alpha_2)}) D X_{(\alpha_2)} + \phi_{12} + \psi_{12}, \quad (159)$$

where

$$\begin{aligned}
 \phi_{12} = & -\frac{1}{2}r_{12}(X_{(\alpha_1)} + X_{(\alpha_2)})\kappa_{(\alpha)} \\
 & -\frac{1}{2}r_{12}S_{(\alpha 44 \beta)}(X_{(\alpha_1)}X_{(\beta_1)} + X_{(\alpha_2)}X_{(\beta_2)} + X_{(\alpha_1)}X_{(\beta_2)}) \\
 & +\frac{1}{4}r_{12}^{-1}S_{(\alpha \beta \gamma \delta)}X_{(\alpha_1)}X_{(\beta_1)}X_{(\gamma_2)}X_{(\delta_2)}, \\
 \psi_{12} = & \frac{1}{2}S_{(\alpha \beta \gamma 4)}(X_{(\alpha_1)}X_{(\beta_1)}X_{(\gamma_2)} - X_{(\alpha_2)}X_{(\beta_2)}X_{(\gamma_1)}).
 \end{aligned} \tag{160}$$

Here $\kappa_{(\alpha)}$ are the components on the Fermi triad of the first curvature vector of C_0 , and the S 's are the components of the symmetrized Riemann tensor of Π -(69). We note the important facts that

$$\phi_{12} = \phi_{21}, \quad \psi_{12} = -\psi_{21}. \tag{161}$$

By changing the numerical suffixes, we can apply (159) to the parts of a circuit [010]. If this circuit starts at $s = 0$, reflects at $s = s_1$, and returns at $s = \bar{s}_0$, we get

$$\begin{aligned}
 s_1 &= r_{01} + X_{(\alpha_1)}DX_{(\alpha_1)} + \phi_{01}, \\
 \bar{s}_0 - s_1 &= r_{10} + s_1Dr_{10} + \phi_{10}.
 \end{aligned} \tag{162}$$

Addition gives the trip-time $\bar{s}_0 = [010]$. Collecting similar results, we have

$$\begin{aligned}
 \frac{1}{2}[010] &= r_{01} + r_{01}Dr_{01} + \phi_{01}, \\
 \frac{1}{2}[020] &= r_{02} + r_{02}Dr_{02} + \phi_{02}, \\
 \frac{1}{2}[030] &= r_{03} + r_{03}Dr_{03} + \phi_{03}.
 \end{aligned} \tag{163}$$

We shall return to these later.

Likewise for the circuit [0120] shown in Fig. 11 we get

$$\begin{aligned}
 s_1 &= r_{01} + X_{(\alpha_1)}DX_{(\alpha_1)} + \phi_{01}, \\
 s_2 - s_1 &= r_{12} + s_1Dr_{12} - (X_{(\alpha_1)} - X_{(\alpha_2)})DX_{(\alpha_2)} + \phi_{12} + \psi_{12}, \\
 \bar{s}_0 - s_2 &= r_{20} + s_2Dr_{20} + \phi_{20}.
 \end{aligned} \tag{164}$$

In the first approximation,

$$s_1 = r_{01}, \quad s_2 = r_{01} + r_{12}, \tag{165}$$

and so we get, on adding (164) together,

$$[0120] = \bar{s}_0 = (r_{01} + r_{12} + r_{20}) + r_{01}Dr_{01} + (r_{01} + r_{12} + r_{20})Dr_{02} \\ + r_{01}Dr_{12} - X_{(\alpha_1)}DX_{(\alpha_2)} + (\phi_{01} + \phi_{12} + \phi_{20}) + \psi_{12}. \quad (166)$$

To get the trip-time for the same circuit described in the opposite sense we have merely to interchange the numbers 1 and 2:

$$[0210] = (r_{02} + r_{21} + r_{10}) + r_{02}Dr_{02} + (r_{02} + r_{21} + r_{10})Dr_{01} \\ + r_{02}Dr_{21} - X_{(\alpha_2)}DX_{(\alpha_1)} + (\phi_{02} + \phi_{21} + \phi_{10}) + \psi_{21}. \quad (167)$$

Noting the symmetries in (161) and that of course $r_{01} = r_{10}$, etc., we obtain, by addition and subtraction of the above equations,

$$\frac{1}{2}[0120210] = (r_{01} + r_{12} + r_{20})(1 + \frac{1}{2}Dr_{01} + \frac{1}{2}Dr_{20}) + \phi_{01} + \phi_{12} + \phi_{20} \\ + \frac{1}{2}r_{01}Dr_{01} + \frac{1}{2}r_{20}Dr_{20} + \frac{1}{2}(r_{01} + r_{20})Dr_{12} \quad (168)$$

and

$$[0120] - [0210] = -(r_{02} + r_{21})Dr_{01} + (r_{01} + r_{12})Dr_{20} \\ + (r_{01} - r_{02})Dr_{12} - (X_{(\alpha_1)}DX_{(\alpha_2)} - X_{(\alpha_2)}DX_{(\alpha_1)}) + 2\psi_{12}. \quad (169)$$

There are similar expressions for $[0230320]$ and $[0310130]$, and for the differences as in (169). Note that the ϕ 's have been separated from the ψ 's.

Subject to the approximations indicated (close world-lines, only slightly curved), the above expressions are general. The control conditions (153) and (154) have not been used. We now impose them. By (153) and (163),

$$T = r_{01} + r_{01}Dr_{01} + \phi_{01}. \quad (170)$$

Since T is constant, Dr_{01} is small and we can neglect the second term on the right. Thus, with the other equations of (163), we have

$$r_{01} = T - \phi_{01}, \quad r_{02} = T - \phi_{02}, \quad r_{03} = T - \phi_{03}. \quad (171)$$

Likewise from (154) and (168) and its fellows, and also (171), we find

$$r_{23} = T' - \phi_{23}, \quad r_{31} = T' - \phi_{31}, \quad r_{12} = T' - \phi_{12}, \quad T' = T\sqrt{2}. \quad (172)$$

Here we have the Fermi lengths of the six edges of the tetrahedron. They differ only slightly from the standard tetrahedron of Fig. 13 with calculable differences.

Under the controls (153) and (154) we have before us a 'nearly rigid' tetrahedron with one vertex fixed, but it is still free to rotate. Let us investigate the effects of this rotation, choosing at any assigned

instant the Fermi triad along the nearly-perpendicular edges of the tetrahedron, so that, to the first order,

$$X_{(\alpha_1)} = (T, 0, 0), \quad X_{(\alpha_2)} = (0, T, 0), \quad X_{(\alpha_3)} = (0, 0, T). \quad (173)$$

Then if $\omega_1, \omega_2, \omega_3$ are the components of the angular velocity of the tetrahedron relative to the Fermi triad, we have for the velocities of its vertices

$$\begin{aligned} DX_{(\alpha_1)} &= (0, \omega_3 T, -\omega_2 T), & DX_{(\alpha_2)} &= (-\omega_3 T, 0, \omega_1 T), \\ DX_{(\alpha_3)} &= (\omega_2 T, -\omega_1 T, 0). \end{aligned} \quad (174)$$

Hence

$$X_{(\alpha_2)} DX_{(\alpha_3)} - X_{(\alpha_3)} DX_{(\alpha_2)} = -2\omega_1 T^2, \quad (175)$$

and two similar expressions.

Leaving this for a moment, we turn to (160) and use (173) to evaluate the ϕ 's. Inserting the results in (171) and (172), we get the following more explicit expressions for the deformation of the tetrahedron:

$$\begin{aligned} r_{01} &= T(1 + \tfrac{1}{2}T\kappa_{(1)} + \tfrac{1}{2}T^2 S_{(1441)}), \\ r_{23} &= T'[1 + \tfrac{1}{2}T(\kappa_{(2)} + \kappa_{(3)}) \\ &\quad + \tfrac{1}{2}T^2(S_{(2442)} + S_{(3443)} + S_{(2443)} - \tfrac{1}{4}S_{(2233)})], \end{aligned} \quad (176)$$

with of course similar formulae obtained by the substitutions

$$1 \rightarrow 2 \rightarrow 3 \rightarrow 1.$$

Note that these expressions do not involve the angular velocity.

We have now to evaluate (169) and its fellows. We turn back to (160) for the ψ 's and use (173) and (175). Thus we get

$$\tfrac{1}{2}([0230] - [0320]) = T[\omega_1 T + \tfrac{1}{2}T^2(S_{(2234)} - S_{(3324)}), \quad (177)$$

and two similar expressions.

Here we have the *Sagnac effect*¹: *the trip-time depends on the sense in which the circuit is described*. If we attach the tetrahedron to the Fermi triad so as to make $\omega = 0$, the Sagnac effect disappears almost entirely. Once more we recognize that Fermi transport corresponds to absence of rotation, although to get rid of the Sagnac effect completely we would have to give the tetrahedron the very small angular

¹ The term $\omega_1 T^2$ represents the Sagnac effect of special relativity; cf. PAULI [1958, pp. 19, 207] and references given there. The S -terms represent the contribution of the gravitational field.

velocity (relative to the Fermi triad) indicated by

$$\omega_1 = -\frac{1}{2}T(S_{(2234)} - S_{(3324)}), \quad (178)$$

and two similar expressions. This is a rotation in a definite sense, determined by the curvature of space-time.

§ 8. A FIVE-POINT CURVATURE DETECTOR

Near the beginning of the preceding section, a goal was set — to design experiments to measure the curvature of the observer's world-line ($\kappa_{(\alpha)}$) and the curvature of space-time (R_{ijkl}). It might be thought that we had reached that goal in (176) and (177), but that is not the case. It is true that (176) contains the curvatures we seek, together with the measurable quantity T , but the Fermi distances r_{01} , r_{23} , etc. also appear. These are mere mathematical constructs — they are not measurable. Likewise in (177), the angular velocity is not measurable,

because Fermi transport is only a mathematical construct.

The fact is that, although the calculations which have been made form an essential basis, we cannot measure the curvatures with an apparatus consisting of only four points — we need at least five. In describing a *five-point curvature detector* (Fig. 14), it is well to omit the mathematical details (which can be supplied by the methods of the preceding section), and state the case clearly in physical terms. It need hardly be said that the apparatus con-

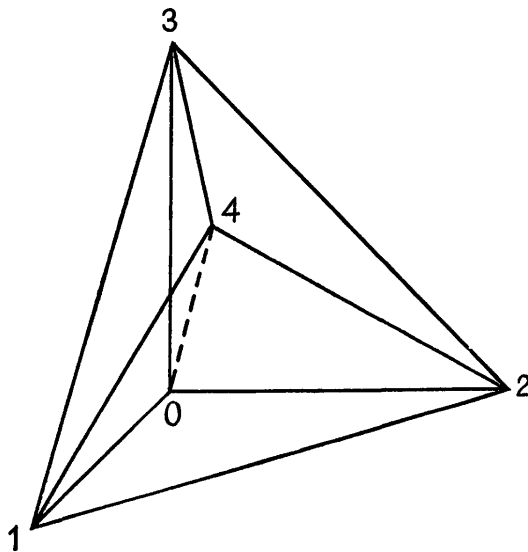


Fig. 14 — A five-point curvature detector.

sidered is a mathematical idealization, with a point source and point mirrors; it bears the same relation to any practical realization of it as the usual textbook description of the principle of the Michelson-Morley experiment bears to the interferometer which they actually employed.

To avoid all possible confusion, one should start, not with a space-picture as in Fig. 14, but with a space-time diagram showing five world-lines. However, that can be supplied by adding one extra world-line to Fig. 11. In passing from such a space-time diagram to a space-picture, we ask: In what space is it drawn? Fig. 12 was drawn

in Fermi space, but that we now reject, because we seek to avoid entanglement with mere mathematical constructs. The only safe plan is to say that Fig. 14 is not drawn in any space at all; it is merely a guide for our thoughts in the discussion of light signals passing between a source 0 and mirrors 1, 2, 3, 4.

Trip-times such as $[010]$, $[0120]$ are measurable. In terms of such measurable trip-times, we define the *optical distances* between the source and the mirrors, and between the mirrors, by formulae of the type

$$\begin{aligned} [01] &= \frac{1}{2}[010], \\ [12] &= \frac{1}{2}[0120210] - [01] - [02]. \end{aligned} \quad (179)$$

Although in practice we might impose controls as in (153) and (154), the principles are better understood if we do not. Then all the optical distances vary with time, and in comparing them we shall deal with the values given for signals leaving 0 at the same time s (measured on the clock at 0 — there is only one clock involved).

It might be thought that, having measured all the ten optical distances at time s , we could make a model in ordinary space out of rods with lengths equal to these optical distances. *The essence of the matter is that such a model could not be made*, except under quite particular circumstances. We could build in all the rods but one, and that last one would not fit. This failure to fit is due to the curvature of the source's world-line and to the curvature of space-time. The fit would be perfect for an apparatus in uniform motion in flat space-time, but for an apparatus on the earth's surface there would be a very minute failure.

To discuss this failure systematically, we recall that in Euclidean 3-space the mutual distances of five points satisfy a certain equation¹. If the five points are labelled as in Fig. 14, and if we use the symbols as in (179) to denote for the moment *Euclidean* distances, this equation reads $D = 0$ where D is the 6×6 determinant

$$D = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & [01]^2 & [02]^2 & [03]^2 & [04]^2 \\ 1 & [10]^2 & 0 & [12]^2 & [13]^2 & [14]^2 \\ 1 & [20]^2 & [21]^2 & 0 & [23]^2 & [24]^2 \\ 1 & [30]^2 & [31]^2 & [32]^2 & 0 & [34]^2 \\ 1 & [40]^2 & [41]^2 & [42]^2 & [43]^2 & 0 \end{vmatrix}. \quad (180)$$

¹ Cf. G. SALMON, *Modern Higher Algebra*, Dublin, 1885, p. 27.

If we insert in this determinant measured optical distances, it will not vanish, and its magnitude is a measure of the curvatures we are seeking. By varying the apparatus in shape and orientation, the information will be increased.

Without going into detailed calculations, we can estimate orders of magnitude. Suppose an experiment is performed and the ten optical lengths recorded. Denote the mean value by T . Let $\delta[04]$ be the difference between the experimental value of $[04]$ and the value found by solving $D = 0$ with the other experimental values inserted. Then it appears from (176) that, as far as orders of magnitude are concerned, we can write the symbolic equation

$$\delta[04] = T^2\kappa + T^3R, \quad (181)$$

where κ is a typical component of the curvature of the source's world-line and R a typical component of the Riemann tensor. On the earth's surface we have roughly

$$\kappa = 3 \times 10^{-8} \text{ sec}^{-1}, \quad R = 3 \times 10^{-6} \text{ sec}^{-2}. \quad (182)$$

If we take

$$T = 1000 \text{ cm} = 3 \times 10^{-8} \text{ sec}, \quad (183)$$

then

$$T^2\kappa = 3 \times 10^{-23} \text{ sec}, \quad T^3R = 8 \times 10^{-29} \text{ sec}. \quad (184)$$

By enlarging the apparatus (increasing T) we might increase these numbers, but it appears that $\delta[04]$ lies far below the level of present accuracy in time-measurement.

However, since conditions on the earth are steady, we might allow the oscillations to be repeated over and over again, so that we would not be dealing with a circuit $[010]$, for example, but with $[010101\dots 10]$. This would enlarge the effect by a factor equal to the number of repetitions. In a week, the number of repetitions would be of the order 10^{13} ; this would bring $T^2\kappa$ up to the level of present accuracy of time-measurement, but T^3R would be too low by a factor 10^{-6} . This is disappointing, but there is a grim satisfaction in pushing the theory of the famous Michelson-Morley experiment to the bitter end.

§ 9. SPECTRAL SHIFT IN A CONTINUUM

A new technique has been reported ¹ for the detection and measurement of spectral shifts for γ -rays with source and receiver both fixed relative to the earth and in other situations too. It is timely to develop the theory of such shifts in order to see what in fact is being measured by the observations.

Mathematically, the continuous is easier to treat than the discrete, and we shall here abandon the method of the preceding Sections in favour of a method in which, in a given space-time, the world-lines of source and receiver are regarded as two world-lines of a congruence specified by 4-velocity $V^i(x)$ ($V_i V^i = -1$). The rays of the radiation are taken to be null geodesics.

Fig. 15 shows two world-lines, C for receiver and C' for source, with a null geodesic $P'P$ passing from source to receiver; O is the foot of the (spacelike) geodesic $P'O$ drawn orthogonal to C . By VII-(230) the spectral shift (red-shift positive) is

$$\frac{v' - v}{v'} = \frac{\Omega_{i'} V^{i'} + \Omega_i V^i}{\Omega_{j'} V^{j'}}, \quad (185)$$

where $\Omega_i, \Omega_{i'}$ are the partial derivatives the world-function $\Omega(P'P)$ with respect to of P, P' , and $V^i, V^{i'}$ are the 4-velocities at those events. We write $OP = s$, $OP' = \tau$ and $\mu^i, \mu^{i'}$ for the unit tangents to OP' at O, P' . The problem is defined by the geometry of space-time, the congruence $V^i(x)$, the unit vector μ^i and the scalar τ ; the plan is to expand in powers of τ .

We note that the situation shown in Fig. 15 was already depicted in Fig. 12 of III-§ 8, and when we make the necessary changes in notation, III-(66) gives

$$s = \tau - \frac{1}{2} \tau^2 \mu_i D V^i + O_3, \quad (186)$$

where $D = \delta/\delta s$; here $\mu_i D V^i$ is evaluated at O .

The quantities $\Omega_{i'} V^{i'}$ and $\Omega_i V^i$ are 2-point invariants, defined for any pair of events P, P' . If we think of sliding P and P' along

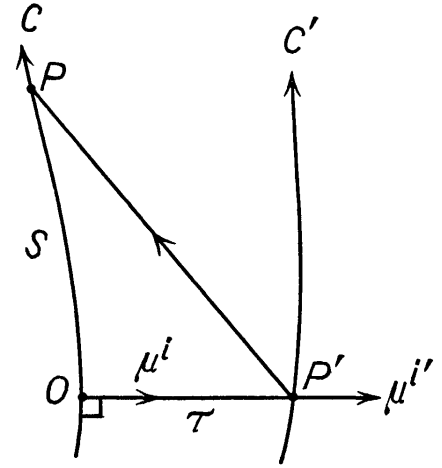


Fig. 15 – Spectral shift in a continuum.

¹ MÖSSBAUER [1958], [1959], POUND and REBKA [1959], SCHIFFER and MARSHALL [1959], MOON [1960]. For a general account, see MARGERISON [1960].

C and OP' independently, these 2-point invariants are functions of s , τ , and we may expand them in power series. Later, we shall substitute for s from (186). Let us write $D = \partial/\partial s$ as already in (186), or $\partial/\partial s$ when applied to an invariant, and $T = \partial/\partial \tau$. Then

$$\begin{aligned}\Omega_i V^i &= [\Omega_i V^i] + s[D(\Omega_i V^i)] + \tau[T(\Omega_i V^i)] \\ &\quad + \frac{1}{2}\{s^2[D^2(\Omega_i V^i)] + 2s\tau[DT(\Omega_i V^i)] + \tau^2[T^2(\Omega_i V^i)]\} \\ &\quad + \frac{1}{6}\{s^3[D^3(\Omega_i V^i)] + 3s^2\tau[D^2T(\Omega_i V^i)] \\ &\quad + 3s\tau^2[DT^2(\Omega_i V^i)] + \tau^3[T^3(\Omega_i V^i)]\} + O_4,\end{aligned}\tag{187}$$

where $[]$ indicates in each case a coincidence limit at the event O . Using II-(69) to evaluate these limits, we obtain

$$\begin{aligned}[\Omega_i V^i] &= 0, \\ [D(\Omega_i V^i)] &= [\Omega_{ij} V^i V^j + \Omega_i D V^i] = -1, \\ [T(\Omega_i V^i)] &= [\Omega_{ij'} V^i \mu^{j'}] = 0, \\ [D^2(\Omega_i V^i)] &= [\Omega_{ijk} V^i V^j V^k + 3\Omega_{ij} V^i D V^j + \Omega_i D^2 V^i] = 0, \\ [DT(\Omega_i V^i)] &= [\Omega_{ijk'} V^i V^j \mu^{k'} + \Omega_{ij'} D V^i \mu^{j'}] = -\mu_i D V^i, \\ [T^2(\Omega_i V^i)] &= [\Omega_{ij'k'} V^i \mu^{j'} \mu^{k'}] = 0, \\ [D^3(\Omega_i V^i)] &= [3\Omega_{ij} D V^i D V^j + 3\Omega_{ij} V^i D^2 V^j + \Omega_{ij} D^2 V^i V^j] \\ &\quad = -D V_i D V^i, \\ [D^2 T(\Omega_i V^i)] &= [\Omega_{ij'} D^2 V^i \mu^{j'}] = -\mu_i D^2 V^i, \\ [DT^2(\Omega_i V^i)] &= [\Omega_{ijk'm'} V^i V^j \mu^{k'} \mu^{m'}] = \frac{2}{3}K, \\ [T^3(\Omega_i V^i)] &= 0.\end{aligned}\tag{188}$$

Here

$$K = -R_{ijkm} V^i \mu^j V^k \mu^m,\tag{189}$$

the Riemannian curvature of space-time for the 2-element defined by V^i and μ^i at O ; all the quantities on the right hand sides in (188) are evaluated at O . By (187) we have then

$$\Omega_i V^i = -s - s\tau\mu_i D V^i - \frac{1}{6}\{s^3 D V_i D V^i + 3s^2\tau\mu_i D^2 V^i - 2Ks\tau^2\}.\tag{190}$$

Changing i into i' , (187) gives the power series for $\Omega_{i'} V^{i'}$, and the

coefficients are as follows:

$$\begin{aligned}
[\Omega_{i'} V^{i'}] &= 0, \\
[D(\Omega_{i'} V^{i'})] &= [\Omega_{i'j} V^{i'} V^j] = 1, \\
[T(\Omega_{i'} V^{i'})] &= [\Omega_{i'j'} V^{i'} \mu^{j'} + \Omega_{i'} V^{i'}|_{j'} \mu^{j'}] = 0, \\
[D^2(\Omega_{i'} V^{i'})] &= [\Omega_{i'jk} V^{i'} V^j V^k + \Omega_{i'j} V^{i'} D V^j] = 0, \\
[DT(\Omega_{i'} V^{i'})] &= [\Omega_{i'jk'} V^{i'} V^j \mu^{k'} + \Omega_{i'j} V^{i'}|_{k'} \mu^{k'} V^j] = 0, \\
[T^2(\Omega_{i'} V^{i'})] &= [\Omega_{i'j'k'} V^{i'} \mu^{j'} \mu^{k'} + 2\Omega_{i'j} V^{i'}|_{k'} \mu^{j'} \mu^{k'} \\
&\quad + \Omega_{i'} V^{i'}|_{j'k'} \mu^{j'} \mu^{k'}] = 2V_{i|j} \mu^i \mu^j, \\
[D^3(\Omega_{i'} V^{i'})] &= [\Omega_{i'j} V^{i'} D^2 V^j] = -V_i D^2 V^i = DV_i DV^i, \\
[D^2 T(\Omega_{i'} V^{i'})] &= [\Omega_{i'j} V^{i'}|_{k'} \mu^{k'} D V^j] = -V_{i|j} \mu^j D V^i, \\
[DT^2(\Omega_{i'} V^{i'})] &= [\Omega_{i'jk'm'} V^{i'} V^j \mu^{k'} \mu^{m'} + \Omega_{i'j} V^{i'}|_{k'm'} \mu^{k'} \mu^{m'} V^j] \\
&= -S_{jmk i} V^i V^j \mu^k \mu^m - V_i V^i|_{km} \mu^k \mu^m \\
&= \frac{1}{3}K + V_{i|j} \mu^j V^i|_k \mu^k, \\
[T^3(\Omega_{i'} V^{i'})] &= [2\Omega_{i'j} V^{i'}|_{k'm'} \mu^{j'} \mu^{k'} \mu^{m'} + \Omega_{i'm'} V^{i'}|_{j'k'} \mu^{j'} \mu^{k'} \mu^{m'}] \\
&= 3V_{i|jk} \mu^i \mu^j \mu^k.
\end{aligned} \tag{191}$$

Thus

$$\begin{aligned}
\Omega_{i'} V^{i'} &= s + \tau^2 V_{i|j} \mu^i \mu^j \\
&\quad + \frac{1}{6}\{s^3 DV_i DV^i - 3s^2 \tau V_{i|j} \mu^j DV^i + s\tau^2(K + 3V_{i|j} \mu^j V^i|_k \mu^k) \\
&\quad + 3\tau^3 V_{i|jk} \mu^i \mu^j \mu^k\} + O_4.
\end{aligned} \tag{192}$$

Adding together (190) and (192), and then substituting for s from (186), we get

$$\begin{aligned}
\Omega_{i'} V^{i'} + \Omega_i V^i &= A\tau^2 + B\tau^3 + O_4, \\
A &= V_{i|j} \mu^i \mu^j - \mu_i DV^i, \\
B &= -\frac{1}{2}\mu_i D^2 V^i - \frac{1}{2}V_{i|j} \mu^j DV^i + \frac{1}{2}K + \frac{1}{2}(\mu_i DV^i)^2 \\
&\quad + \frac{1}{2}V_{i|j} \mu^j V^i|_k \mu^k + \frac{1}{2}V_{i|jk} \mu^i \mu^j \mu^k.
\end{aligned} \tag{193}$$

By (186) and (192) we have

$$\begin{aligned}
\Omega_{i'} V^{i'} &= \tau - C\tau^2 + O_3, \\
C &= \frac{1}{2}\mu_i DV^i - V_{i|j} \mu^i \mu^j.
\end{aligned} \tag{194}$$

Substituting these expressions in (185), the formula for spectral shift becomes

$$\frac{\nu' - \nu}{\nu'} = \tau \frac{A + B\tau + O_2}{1 - C\tau + O_2} = A\tau + E\tau^2 + O_3 \quad (195)$$

$$\begin{aligned} E = B + AC = & -\frac{1}{2}\mu_i D^2 V^i - \frac{1}{2}V_{i|j}\mu^j D V^i + \frac{1}{2}K \\ & + \frac{1}{2}V_{i|j}\mu^j V^i{}_{|k}\mu^k \\ & + \frac{1}{2}(V_{i|jk} + 3DV_i V_{j|k})\mu^i \mu^j \mu^k \\ & - (V_{i|j}\mu^i \mu^j)^2. \end{aligned}$$

The approximation has been carried out to this order to show the effect of the gravitational field; it appears in E . It does not appear in the principal part of the shift, which is

$$\frac{\nu' - \nu}{\nu'} = \tau(V_{i|j}\mu^i \mu^j - \mu_i D V^i). \quad (196)$$

This is easy to interpret. We have

$$V_{i|j}\mu^i \mu^j = \sigma_{ij}\mu^i \mu^j, \quad (197)$$

where σ_{ij} is the rate-of-strain tensor (IV-62). The term $\mu_i D V^i$ is the component of the first curvature vector of the observer's world-line in the direction OP' . If the motion of the continuum is rigid in the Born sense, then $\sigma_{ij} = 0$, and we are left with

$$\frac{\nu' - \nu}{\nu'} = -\tau\mu_i D V^i. \quad (198)$$

If we think of source and receiver carried along with the earth, with the source vertically above the receiver, then μ^i and $D V^i$ both point vertically upwards and $\mu_i D V^i = g$, the usual acceleration due to gravity. Then, assuming rigidity, we have

$$\frac{\nu' - \nu}{\nu'} = -g\tau, \quad (199)$$

a violet shift since negative. In c.g.s. units, this reads $-gh/c^2$ with the height h in cm, c the speed of light in cm sec⁻¹, and $g = 980$ cm sec⁻². For a height of 100 cm, the shift is roughly 10^{-16} .

APPENDIX A

NOTATION

It is well known in human society that the less well-founded on reason a convention is, the harder it is to change and the bitterer the feelings for and against. I have therefore little hope that this reasonable protest against some unreasonable conventions will make any significant changes in them.

EINSTEIN's [1916a] summation convention for repeated suffixes has saved mathematicians and printers from wasting an enormous amount of time in writing and printing useless Σ 's. Against this convention there is nothing to be said (although Levi-Civita was apparently unable to trust it!). But Einstein bequeathed to posterity a more dubious gift — the use of Greek suffixes for the range 1, 2, 3, 4. In this he did not follow RICCI and LEVI-CIVITA [1901] in their fundamental paper on the absolute differential calculus, nor was he followed by PAULI [1921], [1958] in his masterly summary of relativity; these writers used Latin suffixes.

At the present time it is an almost universal convention to use Greek letters for the range (1, 2, 3, 4), or the range (0, 1, 2, 3) which indeed is more generally preferred. It would be cowardly to accept and perpetuate such an unreasonable convention, trivial though the issue may seem. For most of the probable readers of this book, the Latin alphabet is the natural alphabet. The letters are standardized and clear, our typewriters are equipped with them, and they need no special marking for the printer. Why should we use Greek letters, except for special purposes? In this book, Latin suffixes take the values (1, 2, 3, 4) throughout, and Greek suffixes nearly always the subsidiary range (1, 2, 3). Other usage are noted as they occur. The summation convention always operates on the appropriate range, unless deliberately revoked for a special occasion.

In some respects it is unimportant that the metric form of space-time is indefinite. The fact does not obtrude itself in calculating the Riemann tensor, for example. But in other respects it is fundamental,

and it is most unwise to mislead students of relativity by writing down, as a definition of the metric of flat space-time, an equation of the form

$$ds^2 = dx^2 + dy^2 + dz^2 - dt^2 \quad (\text{A-1})$$

or of the form

$$ds^2 = -dx^2 - dy^2 - dz^2 + dt^2. \quad (\text{A-2})$$

This is an undigested heritage from positive-definite days. Whichever of these two expressions we adopt, it implies that ds is real for some displacements and imaginary for others. No serious worker in relativity will be confused by this, for he recognizes (A-1) or (A-2) (whichever he happens to prefer) as a sort of physicists' slang. But why perpetuate such nonsense? What we are concerned with is a quadratic form,

$$\Phi = dx^2 + dy^2 + dz^2 - dt^2, \quad (\text{A-3})$$

or

$$\Phi = -dx^2 - dy^2 - dz^2 + dt^2, \quad (\text{A-4})$$

and $ds = \sqrt{|\Phi|}$, always real. Geometers have been treating the matter thus for over thirty years, and relativity is not such a simple theory to understand that one can afford to muddy the source. Consequently in this book one does not see the curved analogue of (A-1) or (A-2).

But of the curved analogues of (A-3) and (A-4), which is one to choose? Physically, it makes no difference whatever. A mere change in sign of the metric form does not alter the universe it describes. But it does change the signs in certain formulae. It is hard to make a choice on rational grounds, for each of the two forms has its merits. One would prefer that form which is most usually positive, but one cannot go far in relativity without having to deal with both spacelike and timelike displacements. It is true that (A-4) is positive for a displacement which belongs to the world-line of a particle, but it becomes negative if our interest turns to the relationship between the histories of two particles, studying perhaps a possible rigid connection between them.

I have chosen the type (A-3) (signature $+2$, not -2) in this book primarily because I am used to it, having studied Pauli's article long ago and never having found any reason to change except the increasing isolation of anyone who fails to conform to a growing convention. But there is a slight impersonal reason for preferring (A-3). Although

imaginary coordinates have been avoided in this book, there are private occasions on which we simplify complicated formulae by reducing the metric form to a sum of squares, and it is much easier to carry out this mental exercise with *one* imaginary instead of *three*.

There follows a table of conversion from signature $+2$ to signature -2 , but the reader is warned of a further source of confusion! There are two ways of defining the Riemann and Ricci tensors and they differ in sign. The definitions used in this book are stated in I-§ 1.

CONVERSION FROM SIGNATURE $(+2)$ TO SIGNATURE (-2)

Signature $(+2)$	Signature (-2)
$\text{diag}(1, 1, 1, -1)$	$\text{diag}(-1, -1, -1, 1)$
g_{ij}	g'_{ij}
	$g_{ij} = -g'_{ij}$
	$[ij, k] = -[ij, k]'$
	$g = g'$
	$g^{ij} = -g'^{ij}$
$\Gamma^i_{jk} = \{^i_{jk}\}$	$\{^i_{jk}\}' = \Gamma'^i_{jk}$
	$R_{ijkm} = -R'_{ijkm}$
	$R^i_{.jkm} = R'^i_{.jkm}$
	$R_{im} = R'_{im}$
	$R = -R'$
	$G_{ij} = G'_{ij}$
	$G^i_j = -G'^i_j$

DIFFERENTIATION

Partial derivatives are indicated by a comma as in $f_{,i}$, $f_{i,j}$, $f_{ij,km}$.

Covariant derivatives are indicated by a stroke as in $f_{i|j}$, $f_{ij|km}$ and occasionally by a double stroke as in $f_{\alpha||\beta}$.

To simplify the notation, commas and strokes are omitted in situations where confusion is unlikely.

Absolute derivatives are indicated by δ as in $\delta V^i/\delta s$, $\delta V^i/\delta u$, and sometimes by D as in DV^i .

LIST OF PRINCIPAL SYMBOLS

with leading references

A^i = unit tangent vector (I-§ 3) or 4-velocity (III-§ 8).

b = first curvature (I-§ 3).

B^i = unit first normal vector (I-§ 3).

c = second curvature (I-§ 3).

C^i = unit second normal vector (I-§ 3).

$C_i = 0$: coordinate conditions (IV-§ 6).

d = third curvature (I-§ 3).

D = ordinary (or absolute) derivative (I-§ 6, II-§ 12).

D^i = unit third normal (I-§ 3).

E = energy (III-§ 4).

E_{ij} = electromagnetic energy tensor (x-§ 1).

$f(x)$ = frequency function (XI-§ 1).

FC = Fermi coordinates (II-§ 10).

F_{ij}, F_{ij}^* = electromagnetic tensor and its dual (x-§ 1).

$F(x, x')$ = integrand in Fermat's principle (XI-§ 3).

$g = \det(g_{ij})$ (I-§ 1).

g = 'acceleration due to gravity' (III-§ 9).

g_{ij}, g^{ij} = metric tensor and its conjugate (I-§ 1).

$g_{ij'}, g_{i_1 j_2}, g_{i_1 j_2 k_1}, \dots$ = parallel propagator and its covariant derivatives (II-§§ 3, 4).

$g_{abcd} = g_{ac}g_{bd} - g_{ad}g_{bc}$ (I-§ 5).

\bar{g}_{ij} = modified (optical) metric tensor (XI-§ 2).

G = Green's function (I-§ 6).

G_{ij}, G_j^i = Einstein tensor (I-§ 5).

h = Planck's constant (III-§ 7).

$h_{i_1 j_1}, h_{i_1 j_2}, h_{i_2 j_2}$ = deviations of second covariant derivatives of the world-function from the flat values (II-§ 11).

$H_{a'b'}$ = flux of angular momentum (VI-§ 4).

J^i = 4-current (x-§ 1).

K = Riemannian curvature (I-§ 5).

\mathbf{K} = matrix in geodesic deviation (I-§ 6).

$l_{(\alpha)}$ = apparent direction cosines of star (XI-§ 6).

L = finite measure of curve (I-§ 1) and Lagrangian (XI-§ 3).

\mathbf{L} = Lorentz matrix (I-§ 3).

m = (proper) mass (III-§ 3) and mass of star (VII-§ 6).

$M_{a'}$ = flux of 4-momentum (VI-§ 4).

MO = mathematical observations (III-§ 1).

n = refractive index (XI-§ 2).

n^i = unit normal (IV-§ 1).

N^i = numerical vector (IV-§ 1) and unit normal (I-§ 10).

NO = natural observations (III-§ 1).

OC = optical coordinates (II-§ 10).

p = pressure (IV-§ 4).

p^i = 4-momentum (III-§ 3).

$P_j^i, P_{(\alpha\beta)}$ = projection operators (IV-§ 3, XI-§ 6).

$q = \sqrt{-g}$ (X-§ 1).

$Q_{(ab)}$ = matrix in Fermi-Walker transport (I-§ 4).

Q_i, Q_{ij}, \dots = moments of distribution (IV-§ 1).

QC = quasi-Cartesian coordinates (II-§ 8).

r = a curvature coordinate for spherical symmetry (VII-§ 2).

R = curvature invariant (I-§ 5).

R_{ij}, R_j^i = Ricci tensor (I-§ 5).

$R_{ijkl}, R_{jkm}^i, \bar{R}_{12}, \dots$ = Riemann tensor (I-§ 5).

$R_{(abcd)}$ = components on orthonormal tetrad (I-§ 3).

\tilde{R}^{ijkl} = double dual (I-§ 5).

$\bar{R}_{\rho\mu\nu\sigma}$ = Riemann subtensor (I-§ 8).

ds = metric element of space-time (I-§ 1).

dS = element of 3-volume (IV-§ 1) and element of area (VII-§ 6).

S_{ij} = stress tensor (IV-§ 4).

S_{ijkm} = symmetrized Riemann tensor (II-§ 2).

$S_{(\alpha)}$ = 3-vector pointing to star (XI-§ 6).

$S(x', x)$ = Hamilton's principal or characteristic function (I-§ 7).

t = a curvature coordinate ('time') for spherical symmetry (VII-§ 2)

t^{ik} = pseudo-tensor of energy (VI-§ 7).

T_{ij}, T_j^i = energy tensor (IV-§ 1).

$\mathcal{T}^{ij} = \sqrt{-g}T^{ij}$ (VI-§ 6).

u = speed of waves or phase-speed (XI-§ 1).

u^i = 4-velocity of charge (X-§ 1).

$U(x)$ = one-point principal function (I-§ 7).

v = ray-speed or group-speed (XI-§ 2).

$v, v_{(\alpha)}$ = relative speed and velocity (III-§ 7).

v_R = radial speed (III-§ 7).

v^i, V^i = 4-velocity (III-§ 3 and IV-§ 1).

d_2v, d_3v, d_4v = elements of 2-, 3- and 4-volume (I-§ 10).

V = Newtonian potential (III-§ 11).

W_{ij} = aberration tensor (XI-§ 6).

x^i = general coordinates (I-§ 1).

\bar{x}^i = Gaussian coordinates (I-§ 8).

$X^{(a)}, X_{(a)}$ = quasi-Cartesian coordinates (II-§ 8), Fermi coordinates and optical coordinates (II-§ 10).

γ = gravitational constant (IV-§ 5).

γ_{ij} = deviation of g_{ij} from $\text{diag}(1, 1, 1, -1)$ (II-§ 8).

Γ_{jk}^i = Christoffel symbols (I-§ 1).

$\delta_j^i, \delta_{jc}^{ia}, \delta_{jcd}^{iab}$ = Kronecker deltas (I-§ 1, I-§ 5, x-§ 1).

$\delta/\delta u$ = absolute differentiation (I-§ 2).

ε = indicator of curve (I-§ 1).

ε_{ijkm} = numerical permutation symbol (I-§ 5).

$\eta_{(ab)} = \eta^{(ab)}$ = diagonal matrix with elements $(1, 1, 1, -1)$ (I-§ 3).

η_{ijkm}, η^{ijkm} = permutation tensor (I-§ 5).

κ = circulation (I-§ 7) and constant (8π) in field equations (IV-§ 5).

κ^i = first curvature vector (XI-§ 6).

$\lambda_{(a)}^i$ = orthonormal tetrad (I-§ 3).

$\lambda_{(\alpha)}^i$ = frame of reference, usually Fermi-transported (III-§ 5).

Λ = cosmological constant (IV-§ 5).

μ = (proper) density (IV-§ 4).

ν = frequency (III-§ 7).

$\nu = \mu/\rho^2$ (x-§ 2).

$\nu(x, p)$ = distribution function (IV-§ 1).

$\xi_i, \xi_{i|j}$ = Killing vector and tensor (VI-§ 3).

$\rho = \mu + p$ (v-§ 6).

ρ = Gaussian polar coordinate (VII-§ 2) and electrical proper density (x-§ 1).

σ = distance of point from curve (II-§ 10).

$\sigma_{(\alpha\beta)}, \sigma_{ij}$ = rate-of-strain matrix and tensor (IV-§ 3).

$d\sigma^2 = d\theta^2 + \sin^2 \theta d\phi^2$ (VII-§ 2).

τ = trip-time (III-§ 8).

$d\tau^{ij}, d\tau^{ijk} \dots$ = tensor extension of cell (I-§ 10).

θ = polar angle (VII-§ 2).

$\theta_{(\alpha)}$ = direction cosines (III-§ 8).

ϕ = azimuthal angle (VII-§ 2) and phase angle (XI-§ 1).

ϕ_i = 4-potential (x-§ 1).

Φ = fundamental or metric form (I-§ 1).

ω = angular velocity of earth (III-§ 9).

$\omega_{(\alpha\beta)}, \omega_{ij}, \omega_i$ = matrix, tensor and vector of spin or rotation (IV-§ 3).

$\omega(x, y) = 0$: Hamiltonian surface (I-§ 7).

$\omega(x, p) = 0$: medium-equation in optics (XI-§ 2).

$\Omega(P'P), \Omega(P_1P_2), \Omega(x', x)$ = world-function (II-§ 1).

$\Omega_i, \Omega_{i'}, \Omega_{ij}, \Omega_{ij'}, \dots$ = partial and covariant derivatives of world-function (II-§ 1).

$[\Omega_{ij}], [\Omega_{i_1j_2}], \dots$ = coincidence limits (II-§ 2).

APPENDIX B

NUMERICAL VALUES OF SOME PHYSICAL QUANTITIES EXPRESSED IN SECONDS

In the logical structure of relativity, as developed in this book, the fundamental measurement is the measurement of *time*. Length, mass, etc. are derived concepts, and every physical quantity is of dimensions $[T^q]$ where T stand for time and q is some integer or fraction. If $q = 1$, the quantity can be expressed in *seconds* or any other appropriate unit of time. If the 'atomic clock' is accepted as the basic time-measurer, it would be most reasonable to define the unit of time as the period of some standard spectral line, and then define the *second* as some conventional multiple of that period. However it matters little what unit we use, because the important things in physics are the dimensionless ratios of quantities with the same dimensions. Any chosen unit is bound to be inconveniently large for some purposes, inconveniently small for others. We note that for some aspects of celestial mechanics the second is not a bad unit — the radius of the earth's orbit is about 500 sec.

It is very convenient to have all physical quantities expressed in terms of a single unit or powers of it (sec, sec², sec⁻¹, etc.), but such expressions as 3×10^{10} sec or 5.342×10^{-3} sec⁻¹ are clumsy to write, clumsy to print, very clumsy to say in words, and psychologically unsatisfactory, because we like to think in numbers in the range 1 to 100 or thereabouts. To standardize terminology for submultiplies and multiplies of any unit, the International Committee on Weights and Measures at its Paris meeting in 1958 recommended the following prefixes and symbols ¹:

¹ The National Bureau of Standards, U.S.A., has decided to follow these recommendations; cf. Notices Amer. Math. Soc. 7 (1960) 34.

Submultiple or multiple	Prefix	Symbol
10^{-12}	pico (pī'cô)	p
10^{-9}	nano (nā'nc)	n
10^{-6}	micro	u
10^{-3}	milli	m
10^{-2}	centi	c
10^{-1}	deci	d
10	deka	dk
10^2	hecto	h
10^3	kilo	k
10^6	mega	M
10^9	giga (jī'gà)	G
10^{12}	tera (tēr'ā)	T

Thus one writes

$$5 \times 10^9 \text{ sec} = 5 \text{ Gsec}, \quad 3 \times 10^{-9} \text{ sec} = 3 \text{ nsec}.$$

For verbal use, some of the pronunciations are indicated.

However, when one looks at the above table, one sees that the achievement consists in setting up names (the traditional Greek and Latin words with some additions) for the twelve numbers

$$-12, -9, -6, -3, -2, -1, 1, 2, 3, 6, 9, 12,$$

and letters of the alphabet corresponding to them. Since the arabic figures are already international, why should one have to find literal synonyms for them?

It is very difficult to secure international acceptance of a novel proposal, but there is no harm in making one ¹. This proposal takes into consideration international linguistic difficulties, time wasted by writers and printers through the use of a clumsy notation for powers of ten, and the fact that Greek and Latin words are foreign to most scientists and easily forgotten (*ἑκατόν* is indeed one hundred, but *ἕκτος* is the sixth!).

To understand the proposed notation when written, all one has to remember is the technical meaning of two letters, *u* and *d*. To understand it when spoken, the foreigner must learn thirteen words of

¹ The proposal is the result of discussion with colleagues; the up-and-down notation is due to Professor R. J. Duffin.

English, viz.

up, down, point,
one, two, . . . nine, zero.

(He can forget the twelve words shown in the list of the International Committee.)

One writes

$$5 \times 10^9 \text{ sec} = 5 \text{ u } 9 \text{ sec},$$

and one says 'five up nine seconds'; one writes

$$3 \times 10^{-9} \text{ sec} = 3 \text{ d } 9 \text{ sec},$$

and one says 'three down nine seconds'. The up-nine-second is in fact the gigasecond, and the down-nine-second the nanosecond, but their meanings are obvious and one does not have to remember the Greek words for giant and dwarf respectively.

For powers of the unit, the usual sec^2 , sec^3 , sec^{-1} are satisfactory for writing and printing, but it would be easier for foreigners (familiar with the thirteen words mentioned above) if we said 'sec-up-two', 'sec-up-three' instead of 'sec-squared', 'sec-cubed'. As for sec^{-1} , we might say 'sec-down-one' or, better, *inversec*.

All this is submitted as a modest proposal directed towards ease and clarity in scientific communication. There is only one way to find out whether it is a good proposal: try it, and see how it works. The up-and-down notation has not been used in the book; the reduction of all physical quantities to seconds is in itself a sufficiently shocking departure from convention.

The following table shows the values in seconds of a number of physical quantities. The original c.g.s. values have been taken for the most part from the Smithsonian Physical Tables, Ninth Edition, Washington, 1954. Of these original values, the most basic are

$$\text{velocity of light} = c = 2.99776 \times 10^{10} \text{ cm sec}^{-1}$$

$$\text{gravitational constant} = \gamma = 6.670 \times 10^{-8} \text{ g}^{-1} \text{ cm}^3 \text{ sec}^{-2}.$$

The conversion of cm to sec is given by

$$1 \text{ cm} = c^{-1} \text{ sec} = 3.336 \times 10^{-11} \text{ sec}.$$

The conversion of g to sec is given by the relativistic result (cf. IV-§ 5) that

$$1 \text{ g} = \gamma/c^3 \text{ sec} = 2.476 \times 10^{-39} \text{ sec}.$$

Although the interest of these numerical values lies most in orders of magnitude, they are shown to four significant figures to obviate rounding-off errors in calculations. The 'radii of Riemannian curvature' are intended only as a guide to the magnitudes of certain gravitational fields. They are calculated from the formula

$$\text{radius of Riemannian curvature} = (2m/r^3)^{-\frac{1}{2}} \text{ sec},$$

where m is the mass of the body producing the gravitational field and r the distance from its centre, both measured in sec. The curvature of the world-line of a terrestrial observer is what is usually called 'acceleration due to gravity' (g). The radius of curvature is g^{-1} .

The units employed may be described as follows: time is measured in sec, and the units of length and mass so chosen that both the speed of light and the gravitational constant are unity.

QUANTITIES EXPRESSED IN SECONDS,
AND THEIR RECIPROCAL IN INVERSESECONDS

	Value in seconds	Reciprocal in sec ⁻¹ (inversec)
one degree centigrade ¹	3.804×10^{-76}	2.629×10^{75}
electron volt	4.415×10^{-72}	2.265×10^{71}
mass of electron	2.255×10^{-66}	4.435×10^{65}
erg	2.756×10^{-60}	3.629×10^{59}
electronic charge	4.605×10^{-45}	2.172×10^{44}
[Planck's constant] [‡]	1.351×10^{-43}	7.402×10^{42}
gram	2.476×10^{-39}	4.039×10^{38}
electrostatic unit	9.588×10^{-36}	1.043×10^{35}
Ångstrom unit	3.336×10^{-19}	2.998×10^{18}
period of cadmium red line	2.148×10^{-15}	4.655×10^{14}
mass of moon	1.813×10^{-13}	5.516×10^{12}
mass of earth	1.479×10^{-11}	6.761×10^{10}
centimetre	3.336×10^{-11}	2.998×10^{10}
kilometre	3.336×10^{-6}	2.998×10^5
mass of sun	4.920×10^{-6}	2.033×10^5
radius of moon	5.798×10^{-3}	1.725×10^2
mean radius of earth	2.125×10^{-2}	4.706×10
second	1.000	1.000
distance of moon from earth	1.282	7.800×10^{-1}
radius of sun	2.319	4.312×10^{-1}
mean radius of earth's orbit	4.986×10^2	2.006×10^{-3}
radius of Riemannian curvature for earth's field at earth's sur- face	5.697×10^2	1.755×10^{-3}
radius of Riemannian curvature for sun's field at sun's surface	1.126×10^3	8.881×10^{-4}
hour	3.600×10^3	2.778×10^{-4}
[standard density of water] ^{-‡}	3.873×10^3	2.582×10^{-4}
reciprocal of angular velocity of the earth	1.371×10^4	7.292×10^{-5}
sidereal day	8.616×10^4	1.161×10^{-5}
radius of Riemannian curvature for sun's field at earth's surface	3.549×10^6	2.818×10^{-7}

¹ Cf. SYNGE [1957c], p. 44.

	Value in seconds	Reciprocal in sec^{-1} (inversec)
radius of curvature of world-line of terrestrial observer at equa- tor ($=g^{-1}$, $g = 978.05$ cgs)	3.065×10^7	3.263×10^{-8}
sidereal year = light year	3.156×10^7	3.169×10^{-8}
parsec	1.030×10^8	9.709×10^{-9}
estimated age of universe	1.7×10^{17}	5.9×10^{-18}

DIMENSIONLESS QUANTITIES

Weight of one gram	8.079×10^{-47}
Force of attraction between earth and moon	1.631×10^{-24}
Force of attraction between sun and earth	2.927×10^{-22}
Ratio of moon's mass to its radius	3.127×10^{-11}
Ratio of earth's mass to its radius	6.960×10^{-10}
Ratio of sun's mass to its radius	2.122×10^{-6}
Relative velocity of moon and earth	3.397×10^{-6}
Velocity of escape from earth	3.770×10^{-5}
Relative velocity of earth and sun	9.928×10^{-5}
Velocity of light	1
Gravitational constant	1

MISCELLANEOUS

Pressure of one bar = 10^6 dynes cm^{-2} (approx. one atmosphere)	$7.423 \times 10^{-23} \text{ sec}^{-2}$
Density of water (1 g cm^{-3})	$6.668 \times 10^{-8} \text{ sec}^{-2}$

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This Bibliography is intended to be a useful list of references to work on the general theory of relativity and mathematical work closely connected with that theory. It makes no claim to completeness, and could not without making up a substantial volume, for the bibliography of LECAT [1924] ran to close on 200 pages and much work has been done in the thirty-five years since its publication. The decision as to what should be included has inevitably been somewhat arbitrary, but the intention has been to concentrate on what may be called the classical theory, and include only a few leading references to the variants from that theory.

In the case of books, the date follows the author's name. For a journal article, volume, date and page number are shown. Letters inserted after dates are for purposes of reference, distinguishing from one another publications by the same author in the same year. In most cases references to reviews are given, with the following notation:

JF = Jahrbuch über die Fortschritte der Mathematik.

MR = Mathematical Reviews.

SA = Science Abstracts. Section A: Physics.

Z = Zentralblatt für Mathematik.

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